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THE ACCURACY OF LEAST SQUARES SOLUTIONS. PART II: THE STANDARD DEVIATION OF THE ERRORS OF LINEAR EQUATIONS OF CONDITION

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Often we know the results of n observations, $c_1' \ldots c_n'$, such that c_i' may be supposed to have been drawn at random from a normal universe of unknown standard deviation σ (the same for all the observations), and with mean $a_i\xi_a + b_i\xi_b + d_i\xi_d + \ldots + m_i\xi_m$. Here the coefficients $a_i \ldots m_i$ are supposed to be known, while the m quantities $\xi_a \ldots \xi_m$ are unknown. We suppose that $n > m$, and the question that immediately arises is (A): If we know only the values of c_i' and of a_i, b_i, etc., what are the values of the ξ's? The answer to question (A) has been given[1] in a previous paper (referred to henceforth as Part I) in these Proceedings. The next question, perhaps, in order of importance is (B): If we know only the values of c_i' and of a_i, b_i, etc., what is the value of σ?

The classical answer to the problem is well known; it is furnished by the method of least squares. In the notation of Part I,

$$\sigma = s', \tag{10}$$

and this equation has been frequently employed in the practical evaluation of the mean or standard error of the equations of condition of least squares. That equation (10) is not correct has long been known, the resulting defect in the ordinary theory of least squares having been pointed out by Simon Newcomb,[2] who, however, took no steps to correct the theory. While the answer to question (B) is not essential to the answer to question (A), as is apparent from the treatment given in Part I, it may nevertheless be of interest; and it is in any case desirable to present it for the sake of completeness.

Equation (8) of Part I is

$$p(s, a)dsda = \text{const.}'' \times s^{n-m-1}e^{-\frac{(n-m)s^2+a^2|\Delta/\Delta_{aa}|}{2\sigma^2}}dsda \tag{8}$$

giving, in the notation of that paper, the distribution of samples of s' and a'. Integrate over all a. Then

$$p(s)ds = \text{const.}^{\text{IV}} \times s^{n-m-1} e^{-\frac{(n-m)s^2}{2\sigma^2}} ds \qquad (8')$$

is the distribution of s' in the samples. Let $y = \sigma/s$. Then the distribution of y in the samples is

$$p(y)dy = B \times y^{m-n-1} e^{-\frac{n-m}{2y^2}} dy \qquad (11)$$

in which the constant B is defined by

$$B^{-1} = \frac{1}{2}\left(\frac{2}{n-m}\right)^{\frac{n-m}{2}} \Gamma\left(\frac{n-m}{2}\right).$$

Equation (11) gives the chance that σ lies between ys' and $(y + dy)s'$; it is the answer to question (B), and all the properties of the solution follow from it.

Equation (11), and the definition of B, show that under the hypotheses of the first paragraph, $p(y)$ does not depend upon s'. The mode of y is $[(n-m)/(n-m+1)]^{1/2}$; the curve is skew. The mean, $\bar{y}$, is infinite when $n \leqslant m + 1$, and when $n > m + 1$,

$$\bar{y} = \left(\frac{n-m}{2}\right)^{1/2} \Gamma\left(\frac{n-m-1}{2}\right) \Big/ \Gamma\left(\frac{n-m}{2}\right). \qquad (12)$$

The mean value of y^2 is infinite when $n \leqslant m + 2$, and when $n > m + 2$,

$$\overline{y^2} = (n-m)/(n-m-2). \qquad (13)$$

Hence the standard deviation of y about $\bar{y}$, when $n > m + 2$, is

$$\sigma_y = \left[\frac{n-m}{n-m-2} - \frac{n-m}{2}\Gamma^2\left(\frac{n-m-1}{2}\right) \Big/ \Gamma^2\left(\frac{n-m}{2}\right)\right]^{1/2}. \qquad (14)$$

The mean value of σ, on the data of the first paragraph, is $s'\bar{y}$, and the standard error of this value is $s'\sigma_y$.

It is possible to obtain equation (9) of Part I from equation (8) and the results of the present paper. If σ is known, then the distribution of a', obtained by integrating (8), is

$$p(a)da = \frac{1}{\sigma} \text{const.}^{\text{V}}\, e^{-\frac{a^2|\Delta/\Delta a a|}{2\sigma^2}} da,$$

where const.$^{\text{V}}$ does not depend upon σ. If s' is known, the distribution of σ is, by (11),

$$p(\sigma)d\sigma = Bs'^{n-m}\sigma^{m-n-1} e^{-\frac{(n-m)s'^2}{2\sigma^2}} d\sigma.$$

By the law of mutually exclusive events, the distribution of a' when s' is known can be found by integrating the present $p(a)p(\sigma)da d\sigma$ over all σ. Equation (9) is then in fact obtained, showing the consistency of the results given in these papers. Equation (9) is the answer to question (A); and while the answer to question (A) has thus been shown to follow from the answer to question (B), the more direct derivation of equation (9) by the method of Part I is to be preferred.

As in Part I, no use has been made here of inverse probabilities. While the writer has been able to derive the results given in Part I by inverse probabilities, the inverse arguments are considerably inferior to the direct arguments of Parts I and II.

[1] Sterne, *Proc. Nat. Acad. Sci.*, **20**, 565–571(1934)

[2] Simon Newcomb, *Fundamental Constants of Astronomy*, p. 83 (1895).

CHROMOSOME NUMBERS IN CERTAIN RICCIACEAE[1]

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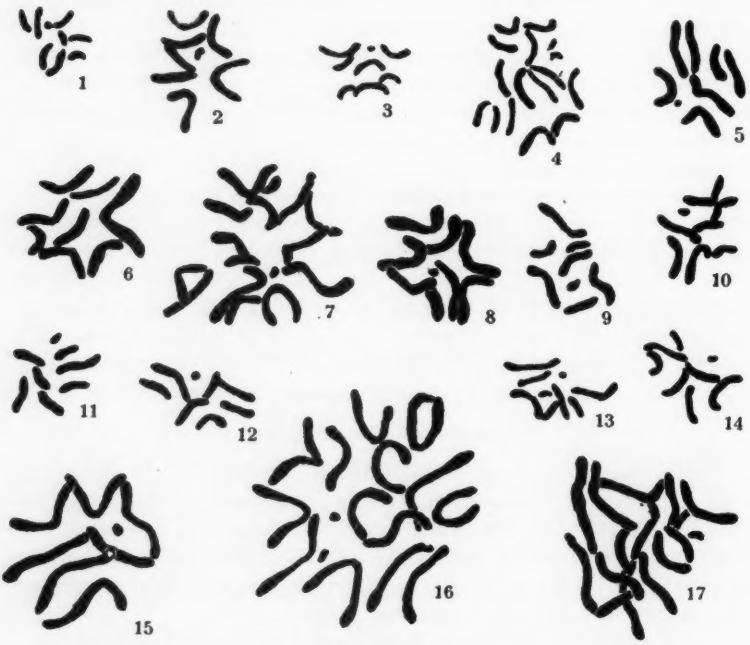
The Ricciaceae have been but little studied cytologically. A preliminary examination of the chromosomes of *Riccia sorocarpa* and *R. Austini* showed that further study of this genus might be interesting. Hence, a more extensive examination of various *Riccias* and of *Ricciocarpus natans* was undertaken. Table 1 shows the chromosome numbers thus far determined. All counts were made from equatorial plates.

Plants of *Ricciocarpus natans* from three sources give the same count, 9 chromosomes in the gametophyte (Fig. 1). These chromosomes are all small, but one is very minute, apparently nearly spherical. Garber[2] and Lewis[3] reported 4 as the gametophytic number for *R. natans*. The small size of the chromosomes and the large proportion of indistinct plates are disadvantageous, but the occasional plate in which the chromosomes are spread out shows 9 very clearly.

The genus *Riccia* is divided into two groups, *Ricciella* and *Euriccia*, on the basis of air-chamber structure. The latter group is much the larger. In the former, counts have been obtained for three species.

In a young antheridium of *Riccia Sullivantii* were found many plates favorable for counting. These invariably contain 7 large chromosomes; many show in addition an eighth very small chromosome (Fig. 2). All the chromosomes (except the smallest) are longer and thicker than any of those of *Ricciocarpus natans*.

Figures in gametophytes of *Riccia crystallina* from Sanford, Fla., show 8 chromosomes, 7 larger and 1 minute (Fig. 3). Mitoses in young sporophytes show 15 or 16 (Fig. 4). Lewis reported only 4 chromosomes in the gametophyte of *R. crystallina*, but here, as in *Ricciocarpus natans*, the chromosomes are all small and clear figures are rare.



FIGURES 1–17

Equatorial plates from various species. Except as otherwise indicated, these plates are from mitoses in young growing regions of thalli. Figure 1, *Ricciocarpus natans*. Figure 2, *Riccia Sullivantii*, antheridium. Figure 3, *R. crystallina*. Figure 4, *R. crystallina*, sporophyte. Figure 5, *R. Curtisii* male, antheridium. Figure 6, *R. Curtisii* female. Figure 7, *R. Curtisii*, sporophyte. Figure 8, *R. Austini*. Figure 9, *R. sorocarpa*, antheridium. Figure 10, *R. glauca*. Figure 11, *R. arvensis*, antheridium. Figure 12, *R. Campbelliana*. Figure 13, *R. californica*. Figure 14, *R. trichocarpa*. Figure 15, *R. Donnellii* male, haploid. Figure 16, *R. Donnellii* male, diploid. Figure 17, *R. Gougetiana* female. All figures ca. 3700 ×.

Riccia Curtisii, a dioicous species, was studied by McAllister[4] who found 8 chromosomes in both gametophytes, one chromosome of each complement being much smaller than the other seven. Males of this species from the U. S. Fish Hatchery southwest of Valdosta, Ga., show likewise

8 chromosomes, 7 large and 1 very small (Fig. 5). Female thalli from Tallahassee, Fla., also show 8, but in this material a very small chromosome is not easily distinguishable (Fig. 6). One chromosome is shorter than the other 7, but it is not so small in comparison to them as is the smallest chromosome in the male. Sixteen chromosomes appear in sporophytic mitoses from the same females (Fig. 7). Here one minute chromosome may be distinguished, but what is probably the second small one is larger.

TABLE 1

CHROMOSOME NUMBERS IN CERTAIN RICCIACEAE

SPECIES	TOTAL NO. IN GAMETOPHYTE	VERY SMALL CHROMOSOMES	SPOROPHYTIC NO.	ORIGIN OF MATERIAL
Ricciocarpus natans (L.) Corda	9	1		Indiana; Columbia Univ.; Hope Lake, Cambridge, Wis.
Riccia				
Ricciella				
R. Sullivantii Aust.	8	1		Tallahassee, Fla.
R. crystallina L.	8	1	16	Sanford, Fla.
R. Curtisii T. P. James	8♂	1		Valdosta, Ga.
	8♀	1	16	Tallahassee, Fla.
Euriccia				
Non-ciliate				
R. Austini Stephani	9	1		State College, Miss.
R. sorocarpa Bisch.	8	1		State College, Miss.; Berkeley, Calif.
R. glauca L., and *R. glauca* L. var. *subinermis* (Lindb.) Warnst.	9	1		Yosemite Valley, Calif.
R. arvensis Aust.	8	1		State College, Miss.
R. Campbelliana M. A. Howe	8	1		Marin Co., Calif.
Ciliate				
R. californica Aust.	9	1 (or 2?)		Sonoma Co., Calif.
R. trichocarpa M. A. Howe	8	1		Placer Co., Calif.
R. Donnellii Aust.	8♂	1		
	16♂	2		Sanford, Fla.
R. Gougetiana Mont.	18♀	2		Columbia Univ.

Black's[5] count of 8 for *R. Frostii*, and Heitz's[6] of 7 + 1 for a *Ricciella* from Algiers, together with the numbers here reported, may indicate that 8 is a characteristic number for the *Ricciella* group. Heitz's count of 14 (−15) for *R. fluitans* suggests that it may be a polyploid.

The *Euriccia* group is subdivided into two, one with naked thallus

margins and one in which the thallus margins bear setae or cilia. Howe[7] lists 13 non-ciliate and 5 ciliate native North American species. Five species from the former and four from the latter sub-group are here considered.

Thalli and antheridia of *Riccia Austini* from Mississippi show clearly 9 chromosomes, including one very small one (Fig. 8). In some favorable figures the small chromosome is somewhat elongated.

Heitz gives as the number for *R. sorocarpa* 8(?) + 1. Clear plates in thalli and antheridia of my material of this species, both from Mississippi and from California, show 8 chromosomes, including one minute one (Fig. 9). The seven larger chromosomes are smaller than those of *R. Austini*, *R. Sullivantii* or *R. Curtisii*, but larger than those of *R. crystallina*. In the haploid chromosome complement of *R. sorocarpa* certain chromosomes may be identified. In addition to the very small chromosome there are 2 short, 2 of medium length and 3 long ones.

For *R. glauca* Beer[8] gave 7 or 8 as the haploid number. Heitz found 8 large and 1 small chromosome in this species. Wentzel[9] reported 8 chromosomes. In the present study, thalli of *R. glauca* show 9 chromosomes, 8 large and 1 small (Fig. 10). The material studied was a mixture of *R. glauca* and its variety *subinermis* from Yosemite Valley, Calif.

The 8 chromosomes of *R. arvensis* (Fig. 11) are very similar to those of *R. sorocarpa*.

In plates from thalli of *R. Campbelliana* 8 chromosomes are present (Fig. 12). The smallest chromosome is very minute but definitely present.

Of the native American ciliate *Riccias* three have been available so far. *R. Gougetiana*, a ciliate species from Algeria, was obtained from the Columbia University greenhouses.

In *R. californica* (Fig. 13), 2 of the 9 chromosomes are much smaller than the other 7. One of these 2 is very small, apparently corresponding to the smallest one in the other *Riccias*, while the second one is two or three times the length of the first, but much shorter than the remaining 7.

In *R. trichocarpa*, 8 chromosomes appear on plates in the thalli (Fig. 14). One of these is very small.

Only male gametophytes have so far been identified in *R. Donnellii* from Sanford, Fla. In this species the chromosomes are large and the plates are diagrammatically clear. Of five thallus tips of this material which yielded figures when sectioned, three were found to have 8 chromosomes as the characteristic number (Fig. 15). Two of these thalli were male and on the third no sex organs were found. The other two thalli, both male, presented plates with 16 chromosomes (Fig. 16). The haploid males have 1 small chromosome among the 8. The diploid males have 2 small chromosomes and apparently a double set of the 7 larger ones.

Female gametophytes of *R. Gougetiana* present a large number of

chromosomes (Fig. 17), not, however, so diagrammatically arranged as in *R. Donnellii*. In the less crowded plates 16 large chromosomes and usually 1, sometimes 2 small ones may be counted. The count here is 18, and these plants are probably diploids in a line with 9 as the basic number. The gametophytes of *R. Donnellii* and *R. Gougetiana* are large and thick, features which might be correlated with the large chromosome size and also with diploidy. In *R. Donnellii* the cells of the diploids are larger than those of the haploids. It has not yet been possible definitely to correlate external features of this latter plant with haploidy or diploidy.

Heitz cites an additional *Euriccia, R. Lesquereuxii*, with 9 chromosomes. Lorbeer [10] found 8 in *R. Bischoffii*.

Oxymitra is the third genus of the *Ricciaceae*. Heitz gives 8 (+1?) as the haploid number in *Oxymitra pyramidata*.

Haupt[11] found only one basic number, 9 chromosomes, in the genus *Marchantia*. In the genus *Riccia* and the family *Ricciaceae* two basic numbers seem to occur, 8 and 9. All the species here reported have either 8 or 9 chromosomes or twice these numbers. In each species one of the 8 or 9 has been found to be much smaller than the other 7 or 8. In the diploids two very small chromosomes are found. Heitz identifies the small ("*m*-") chromosome as a sex-chromosome. In the *Riccias* here reported I have found no evidence as to the function of the smallest chromosome or chromosomes.

[1] Work done with the aid of grants from the University of Wisconsin Research Fund and the Wisconsin Alumni Research Foundation.

[2] Garber, J. F., *Bot. Gaz.*, **37**, 161–177 (1904).

[3] Lewis, C. E., *Ibid.*, **41**, 109–138 (1906).

[4] McAllister, F., *Bull. Torrey Bot. Club*, **55**, 1–10 (1928).

[5] Black, Caroline A., *Ann. Bot.*, **27**, 511–532 (1913).

[6] Heitz, E., *Abhandl. des Naturw. Vereins zu Hamburg*, **21**, 48–58 (1927).

[7] Howe, M. A.; *No. Am. Flora*, **14**, part 1, 7–27 (1923).

[8] Beer, R., *Ann. Bot.*, **20**, 275–291 (1906).

[9] Wentzel, Richmut, *Marburg Dissertations*, **15**, No. 7 (1929).

[10] Lorbeer, G., *Zeit. Indukt. Abst. Vererb.*, **44**, 1–109 (1927).

[11] Haupt, Gertraud, *Ibid.*, **62**, 390–428 (1932).

REVERSIBLE COAGULATION IN LIVING TISSUE. XIII

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In an earlier paper,[2] it was pointed out that caffeine should not keep one awake when its stimulating action is counteracted by sodium rhodanate. One of the main actions of caffeine[3] is to increase the reflex excitability of the central nervous system from above downward, thus among other things giving rise to insomnia. On the other hand, sodium rhodanate is a sedative. Examination of tables 1 and 2, for moderate doses, reveals that any antagonism between these two drugs is most easy to explain from the point of view of their effects on the protein colloids of the nervous system, since the two drugs show opposite actions only on spinal cord and general excitability and on protein colloids.

The differences in the effects of sodium rhodanate and caffeine thus are found in their actions on the nervous system and on protein colloids. In addition, caffeine clearly produces insomnia in some people when taken in the form of coffee, although Laird[12] thinks that psychic stimulation from other causes is mainly responsible. On the other hand, sodium rhodanate tends to reduce the reflex excitability of the nervous system and allows sleep, due to its action in peptizing the protein colloids of the sensory nerves from the slightly agglomerated state, representing stimulation, back to normal. Sodium bromide, which is similar to sodium rhodanate except that it is a weaker peptizing agent for proteins, among other drugs, is recommended[13] for the treatment of acute caffeine poisoning.

With these facts in mind we have conducted clinical experiments on the antagonistic action of these two drugs as represented by the response of insomnia induced by caffeine to the peptizing action of sodium rhodanate. The cases that were accepted for the experiment were those of people who through a great deal of experience with coffee were thoroughly versed in its effect on their sleep. We aimed to give more than enough sodium rhodanate to each patient, except where indicated, to counteract completely the effect of the coffee that they took. Each patient was instructed not to alter, or did not alter radically, his daily habits during or just before the experiment. All doses of sodium rhodanate were given in a glass of water. Sodium rhodanate is not a habit-forming drug.

Case 1. C. A., male, age 54, health good except for heart disease. Coffee habits: Always takes one and one-half cups of coffee at night, one to two cups at breakfast and one cup at noon. Sleeps well only if he retires after 10 P. M. shortly following a light lunch. The night previous to the experiment without altering his coffee habits he retired earlier than

usual and omitted the light lunch; he did not sleep well. The day of the experiment the patient drank the usual amount of coffee, took ten grains of sodium rhodanate at 7 P. M., omitted the light lunch, retired at 9 P. M. and slept better than for several nights preceding the experiment.

TABLE 1

DRUG	METABOLISM	PULSE RATE	DIURESIS	BLOOD PRESSURE	PARASYMPATHETIC STIMULATION	RATE OF RESPIRATION
Caffeine	Increased[4]	Decreased	Strong	Sl. rise	Positive[6]	Increased
NaSCN	No effect[5]	Decreased[6]	Slight	Lowered	Positive[6]	Increased
Coffee		Increased[7]		Lowered[7]		Increased[7]

TABLE 2

DRUG	SPINAL CORD EXCITABILITY	GENERAL EXCITABITITY	SKELETAL MUSCLE	EFFECT ON PROTEIN COLLOIDS	EFFECT ON FATIGUE
Caffeine	Increased	Increased[8]	More lactic acid[14]	Coagulates[9,10,11]	Decreases
NaSCN	Decreased	Decreased	present with both drugs	Peptizes	No change

Case 2. N. F. M., male, age 26, health excellent. Coffee habits: Never uses coffee, being quite sensitive to it as shown by the fact that he sleeps very poorly after taking one cup at dinner time. For two nights prior to the experiment, he took one cup of coffee at dinner. On the second of these nights he reported that he was very restless all night; he complained of an itching sensation and sudden awakening at frequent intervals. At 7:15 P. M. on the third night in succession that he had taken one cup of coffee, he was given fifteen grains of sodium rhodanate. The next day he reported that he had not slept any longer than on the preceding night but that he had rested quietly all night, not being psychically and nervously upset by the insomnia as he had been on the previous nights. The next night, after again taking a cup of coffee at dinner time, he was given another fifteen grains of sodium rhodanate. In the morning he reported that he did not sleep as well as on the previous night and that he was restless and psychically upset as a result of the insomnia. That meant that too much sodium rhodanate had been given, resulting in a slight over-peptization of the sleep center colloids. It has been a common experience in the general work with sodium rhodanate to over-dose the patient slightly; as a consequence the patients have invariably not slept for one night during which time they were not restless, nervous or psychically upset. When the over-dosage has been continued for another night, the patients have continued to sleep poorly; but in contra-distinction to the first night, they have been nervous, restless and psychically upset by the insomnia. Clearly, it is not a usual experience for a man to go most of a night without sleeping and yet not be upset during the night by the experience. So, this patient was very sensitive to sodium rhodanate as

well as to caffeine. On the third night of the experiment, the patient again took a cup of coffee at dinner time; following that he was given seven and one-half grains of sodium rhodanate. In the morning he reported that he had slept well. Therefore, after adjusting the dosage of sodium rhodanate, the antagonism to caffeine was clearly demonstrated. The patient reported that upon taking coffee regularly a tolerance, from the point of view of insomnia, does not develop.

Case 3. A. P. M., male, age 45, health excellent. Coffee habits: One cup of coffee in the morning and a small cup at noon. Occasionally he takes a cup of coffee at 4 P. M. which keeps him awake invariably if he does not take lacto-dextrin at bedtime. Ordinarily a cup of coffee at supper time keeps him awake. On the night of the experiment, he took a cup of strong coffee at supper time followed by fifteen grains of sodium rhodanate. In the morning he reported that he had slept well during the night. He had not taken lacto-dextrin.

Case 4. W. D. B., male, age 67, health excellent. Coffee habits: One cup of coffee daily, in the morning. Coffee taken at night regularly produced insomnia. The first experiment was done two years ago. After one cup of coffee at dinner time, he took seven and one-half grains of sodium rhodanate. In the morning he reported that he had not slept well. A few weeks later, he again took one cup of coffee at dinner time and followed it with fifteen grains of sodium rhodanate. The next morning he reported that he had slept well. Up to this point this case shows two things. In the first place, the sleep was not brought about by the psychic effect of taking the sodium rhodanate, because he did not sleep well on seven and one-half grains. Secondly, the dosage of sodium rhodanate that is required to counteract one cup of coffee varies from individual to individual. Two years later, after having taken seven and one-half grains of sodium rhodanate a week in the interim, the patient took three grains of sodium rhodanate after having taken one cup of coffee at dinner time. In the morning he reported that he had not slept well. A few days later, he took seven and one-half grains of sodium rhodanate following one cup of coffee at dinner time. The next morning he reported that he had slept well, not suffering from insomnia. Another experiment was performed in which the patient took seven and one-half grains of sodium rhodanate on one night and then a cup of coffee at dinner time on the next night. The following morning he reported that he had again slept well. The second set of experiments again shows the absence of a psychic effect from taking the sodium rhodanate. Furthermore, they show either that the sensitivity of the patient to caffeine has decreased or that the sensitivity to sodium rhodanate has increased. In this case, it takes just as much sodium rhodanate to counteract insomnia which is not caused by coffee now as it did when the first experiment was done. Therefore, the

sensitivity to caffeine has decreased. This is considered to be due to a decrease in the readiness with which the protein colloids of the nervous system agglomerate as a result of the administration of sodium rhodanate over a long period of time.

Case 5. A. J. M., male, age 42, health excellent except for occasional kidney stones. Coffee habits: Takes two cups of coffee in the morning and one cup at noon. Usually one cup of coffee at night causes insomnia. The first experiment was made ten months prior to the second one. The patient took one cup of coffee at dinner, followed by fifteen grains of sodium rhodanate. In the morning the patient reported that he had had a good sleep and felt relaxed. The patient reported a like experience shortly afterward. Ten months later, he was given two grains of sodium rhodanate in the same volume of solution. He was told that the solution contained the same amount of sodium rhodanate as that that he had had previously but that the taste had been disguised in a special way. This was taken after two cups of reasonably strong coffee. He reported in the morning that he had not slept as long as usual and that his sleep was unaccountably disturbed. He further reported that he was tired when he retired. This experiment shows again that a certain minimum dose of sodium rhodanate is necessary to counteract the effects of caffeine and that the psychic effect of taking the sodium rhodanate can be ruled out. Of course, the small amount of sodium rhodanate was given purposely in order to show that there was not a predominant psychic effect. Before this last experiment was made, the patient was convinced as a result of his previous experiences that sodium rhodanate counteracts the effects of coffee in producing insomnia. Kidney stones were not a complicating factor during these experiments.

Case 6. M. S. K., male, age 39, health excellent so far as known. Coffee habits: Two cups of coffee at breakfast, occasionally one or two cups at noon. He reported that if he takes two cups of coffee at noon he feels lethargic during the afternoon. He rarely takes a cup of coffee at night; and that usually keeps him awake unless he goes to bed very late. On the night of the experiment, he took one and three-quarters cups of coffee at 6:40 and fifteen grains of sodium rhodanate ten minutes before going to bed at 10:30. In the morning he reported that contrary to his usual experience with one cup of coffee and an early retiring hour, he slept well.

Case 7. H. B., female, age 40. Coffee habits: Unable to drink coffee due to the insomnia which it regularly produces. She drank a cup of coffee with the evening meal and followed that with fifteen grains of sodium rhodanate. She reported that she slept well that night.

Case 8. Mrs. G., age 42. Health: Thyroidectomy done fifteen years ago for exophthalmic goiter; still hyperthyroid with some exoph-

thalmus; neurasthenic and hysterical. Coffee habits: Cannot sleep if she drinks coffee, drinks decaffeinized coffee regularly. Immediately prior to the start of the experiment she had run out of the decaffeinized coffee and so had used regular coffee for three days. During this time she had suffered from insomnia and had come in for relief. She was given fifteen grains of sodium rhodanate and took another cup of ordinary coffee at dinner. The next morning she reported that she had slept well.

Case 9. Mrs. R. claims that coffee always disturbs her sleep. She took one cup of coffee at dinner time and followed that with fifteen grains of sodium rhodanate. In the morning she reported that she had slept well, despite the coffee.

Case 10. J. P., male, age 51, general health fair except for a slight cold at the time of the second experiment. Coffee habits: Drinks coffee regularly in the morning. He drinks tea infrequently. He does not sleep well at night after a cup of coffee. The first experiment was made a month or so prior to the second. The first time he took approximately ten grains of sodium rhodanate in the evening following a cup of coffee at dinner. He slept well that night. At noon of the day of the second experiment, he took ten grains of aspirin to relieve a headache due to considerable eye work. In the afternoon he took a cup of tea following a walk. He reported that he periodically awakens at 4 to 5 A. M., drinks water and goes back to sleep. He took a small cup of coffee at dinner and eleven grains of sodium rhodanate upon retiring at midnight, at which time he was fairly tired. He slept nine hours without interruption. He reported that he usually sleeps somewhat less than that.

More controls would have been desirable; but the patients were volunteers and were not enthusiastic over sleepless nights. The main object of the study was to demonstrate the antagonism between coffee and sodium rhodanate and determine the best probable dose to start with. This seems to be fifteen grains of sodium rhodanate in a glass of water. Three of the patients were trained observers.

[1] M. D.

[2] Bancroft and Rutzler, *Proc. Nat. Acad. Sci.*, **19**, 73 (1933).

[3] Sollmann, *A Manual of Pharmacology*, 273 (1932).

[4] Means, et al., *Arch. Int. Med.*, **19**, 832 (1917).

[5] Bancroft, Gutsell and Rutzler, *J. Phys. Chem.*, **36**, 2011 (1932).

[6] Takács, *Z. ges. exptl. Med.*, **50**, 432 (1926); Sollmann, loc. cit., p. 279.

[7] Sollmann, loc. cit., p. 270.

[8] Pavlov, *Conditioned Reflexes*, 127 (1927).

[9] Jacoby and Galowinski, *Arch. exp. Path. Pharm.*, **59**, Suppl. 286 (1908).

[10] von Fürth, *Ibid.*, **37**, 389 (1896).

[11] Sollmann, loc. cit., p. 283.

[12] Laird and Muller, *Sleep*, 171 (1930).

[13] Sollmann, loc. cit., p. 288.

[14] Sollmann, loc. cit., p. 276; Selter, Hoppe-Seyler's *Zeit. Physiol. Chem.*, **165**, 18 (1927).

GENETICS OF HERMAPHRODITISM IN DROSOPHILA VIRILIS

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Studies of intersexuality in *Drosophila melanogaster* (Bridges)[1] and in *Lymantria dispar* (Goldschmidt)[2] indicate that: (1) each individual possesses genes or potentialities of both sexes, (2) the ultimate sex of an individual is determined by the quantitative relation or the balance between male and female sex factors, (3) at any given time potentialities of one sex are overbalancing those of the other, (4) the reaction of a given sex can be overtaken by that of the other sex with the ultimate result of sex reversion or intersexuality.

These principles of the genic balance theory have been successfully applied in genetical studies of sex. However, they were found to be not quite sufficient in those species in which hermaphroditism commonly occurs.

In the present paper hereditary non-functional hermaphroditism in *Drosophila virilis* will be briefly described, which may help to fill the gap indicated above.

Origin.—The sex-intergrades here described were first observed in 1933 in *sv* stock, which originated in 1930–31 in one of the cultures received from Professor T. Komai, Kyoto, Japan, and kept in the laboratory. By inbreeding normal flies from that stock, 40 cultures were obtained which produced 1993 males, 1516 females and 489 sex-intergrades, while 41 cultures produced 2088 males, 1191 females and 962 sex-intergrades. The number of normal males, females and sex-intergrades in the first set of cultures approaches a ratio of 4:3:1, and in the latter set of cultures a ratio of 2:1:1. This is to be expected, if sex-intergradation is caused by an autosomal recessive gene, which has an effect on females only (Lebedeff).[3]

Linkage.—To locate the gene, linkage tests were performed with rounded (*R*), garnet (*G*) and clipped (*Cl*) dominant genes of the second, third and fourth chromosomes, respectively. The results showed that the gene, called intersex (*ix*) is linked with the third chromosome gene, garnet (*G*). By means of a three point test its locus was found to be at 101.5.

Sex-Linked Genes and Sex-Intergrades.—In order to demonstrate that the sex-intergrades of all morphological types are not gynandromorphs, which they often resemble (Dobzhansky)[4] and in order to show that they are of *XX* constitution, females homozygous for either *v, f, si, mt,* sex-linked genes, and heterozygous for *ix* were crossed to males from stocks carrying the *ix* gene. From these crosses 934 males which carried one of the indicated sex-linked factors, 800 wild type females and 397 wild type sex-inter-

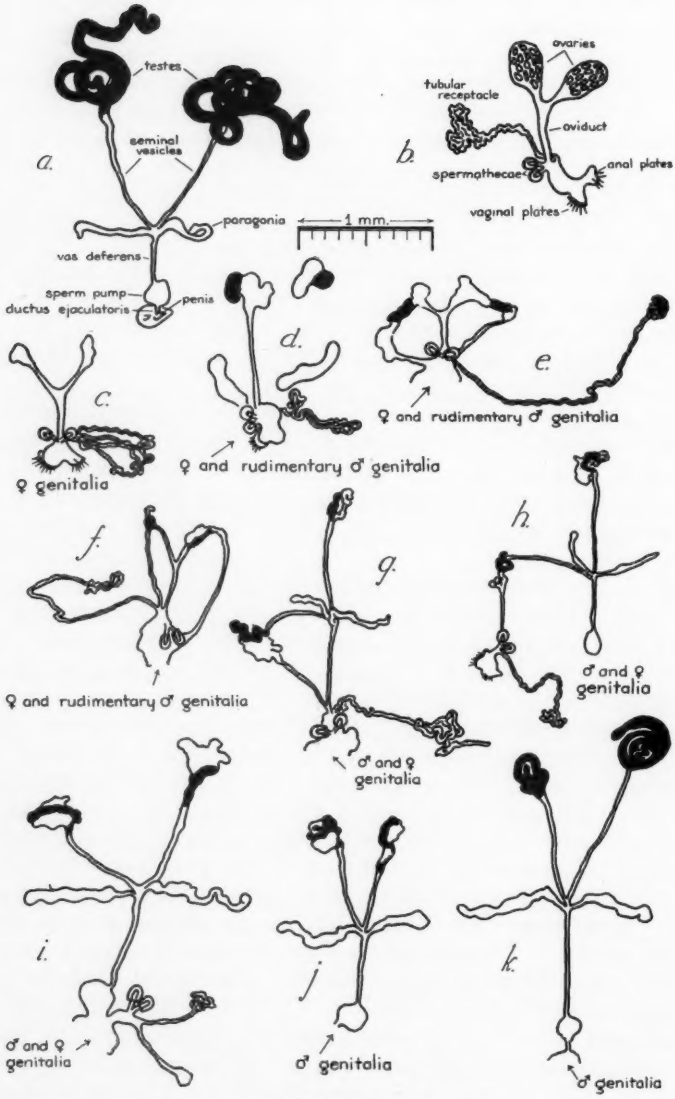

FIGURE 1

a, Reproductive organs of normal male; *b*, of normal female (parovaria omitted); *c–f* of sex-intergrades of morphological type 1; *g–i* of type 2; *j* of type 3; *k* of type 4.

grades of all morphological types were obtained. (The deficiency in males probably is due to sex-linked semi-lethals.)

Morphology of Sex-Intergrades.—Morphologically sex-intergrades exhibit high variability in both external and internal structures. The extreme female type is very similar to normal females in appearance and possesses normal female reproductive organs, with the exception that the ovaries are undifferentiated. On the other hand, the extreme male-type sex-intergrades are simiiar to the normal males, except that they possess testes which are underdeveloped and which produce no sperm. These two extremes are connected by a series of hermaphrodites with variously developed male and female reproductive organs. However, they can easily be grouped into four rather distinct classes.

The first class, the most variable, is composed of females in which male sexual organs may be developed in various degrees from the appearance of embryonic testes to an almost complete male system. However, even in extreme cases both male and female gonads, as well as male reproductive system, and in particular the male external genitalia, are poorly developed (Fig. 1, *c–f*).

The second class is composed of hermaphrodites which have poorly developed male and female gonads, but which have sexual ducts and external genitalia of both sexes well developed. As in the sex intergrades of the first group, their anal plates have characteristics of both sexes, and their abdomens are female-like (Fig. 1, *g–h*).

The third class has well developed male gonads, sexual ducts and genitalia. In addition, the flies have a pair of embryonic ovaries, which are attached to the testes. The testes seem to contain spermatogonia most of which are degenerating, while the ovaries have oöcyte-like cells. They still retain the female shape of the abdomen, but the anal plates are male-like (Fig. 1, *j*).

The fourth class is composed of females that are completely reverted into males. There is no trace of femaleness, except in a few instances, in which a rudimentary fifth (female) sternite is present. The testes, however, rarely reach normal size, and contain mostly spermatocytes, often degenerated. They are sterile, as are all the other types (Fig. 1, *k*).

Sex-Intergrade Modifiers.—After but few generations of selection, three lines, designated as 1, 3 and 4 according to the morphological types described, were isolated each producing only one type of sex-intergrades. Line 3 is an exception, since about twenty per cent of sex-intergrades here are hermaphrodites of type 2. This fact indicates that the degree of sex-intergradation is genetical in origin.

To test the genetical relations between different morphological types, Lines 1, 3 and 4 were outcrossed to the same unrelated stock. In the F_2 of the Line 1 outcross all four types of sex-intergrades appeared. In the

F_2 of the Line 3 outcross sex-intergrades were of type 3 and 4 approximately in 3:1 ratio. In the F_2 of the Line 4 outcross only type 4 intergrade appeared.

These results suggest that females homozygous for *ix* gene are reverted into males, but sterile (type 4). The other described types of sex-intergrades are caused by modifying factors, which retard sex-reversion. The presence of one dominant modifier together with *ix* gene probably results in the development of hermaphrodites of type 3. The relation between type 1 and 3 of sex-intergrades is more complicated, probably involving two or more modifiers. Studies on this phase of the problem are not yet completed.

Discussion.—Morphological and genetical studies of sex-intergrades indicate that they are gametic females, in which the male reproductive system is developed, as a result of the action of a single autosomal gene—*ix*. The time when male development begins is determined by modifying factors, and this in turn, decides the morphological type of sex-intergrade. It is assumed that the early activity of the *ix* gene will suppress the female tendencies and the result will be an almost complete sex-reversion—intergrades of types 3 and 4. Late action of the *ix* gene in the ontogeny of the female will result in development of almost normal females. However, if *ix* starts to act at a certain moment, presumably at the time of the development of the imaginal discs of sexual organs, male and female organs develop simultaneously, resulting in production of hermaphrodites.

According to the assumption, derived from the genic balance theory, the equilibrium between male and female determining genes or substances ($M = F$) would bring about hermaphroditism. The experiment described above indicates that in the case of *D. virilis* this condition is reached in the gametic females homozygous for an autosomal gene *ix*, when its action is delayed by the presence of one or more modifiers.

Summary.—The third chromosome recessive gene, called intersex (*ix*) was found to cause reversion of homozygous females into males, which are sterile.

The incomplete reversion brought about by modifying factors results in production of hermaphrodites. It is assumed that these modifiers determine the time of activity of the *ix* gene and thus determine the degree of hermaphroditism.

[1] Bridges, C. B., *Sex and Internal Secretion*, pp. 55–93 (1932).
[2] Goldschmidt, R., *Bibliog. Genet.*, **11**, 1–185 (1934).
[3] Lebedeff, G. A., *Amer. Nat.*, **68**, 68–69 (1934).
[4] Dobzhansky, Th., *Roux' Arch. Entwicklm.*, **123**, 719–746 (1931).

ORGANIZATION OF SALIVARY GLAND CHROMOSOMES IN SCIARA IN RELATION TO GENES[1]

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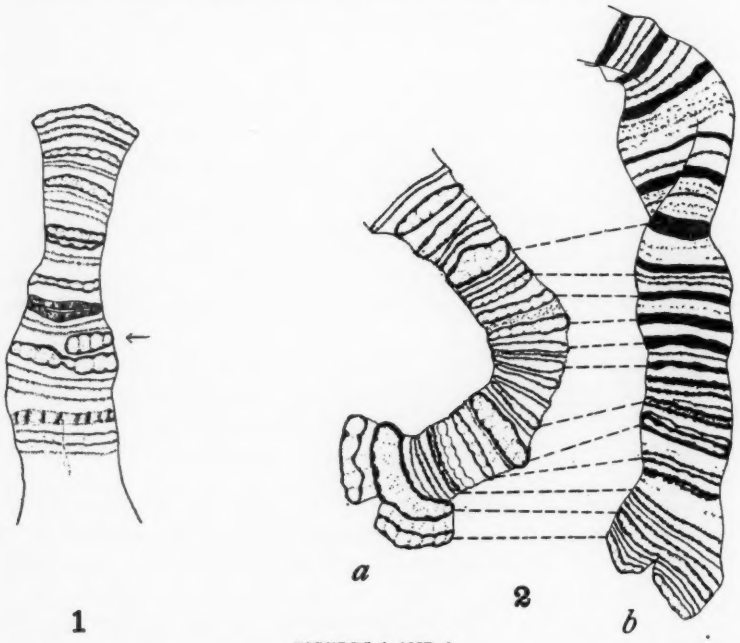
Following the discovery by Heitz and Bauer[2] that the conspicuous and highly organized "spireme" in salivary gland nuclei of Diptera is not continuous, but is composed of chromosomes fused side by side in pairs, much attention has been centered on these structures. Painter[3] has shown convincingly that in *Drosophila* the "discs" or "bands" producing cross striations in the chromosomes bear a definite relation to gene loci, and Bridges[4] and Koltzoff[5] have suggested that the "granules" often visible in the bands represent, in some manner, the individual genes. The latter suggestion is apparently based on the assumption that the two original threads going to make up the chromosome have divided repeatedly, so that each cross band or disc is compound and consists of a series of genes derived from the original pair (or from a single gene in the case of the unpaired *X* chromosome).

The "granules" have been described as forming series of spiral rows along the chromosome, and Koltzoff has noted thread-like lines connecting the granules in these rows. On this interpretation the chromosome is composed of a series of granulated threads, like strings of beads, arranged spirally.

Since the salivary gland chromosomes in *Sciara* are very favorable for observation, we have been making a careful study of their structure in the hope of throwing further light on the problem of chromosome organization, in connection with a study of the sex chromosomes in our material. Some of the results of this study have been noted elsewhere (in press). The present paper is concerned especially with a condition in which one chromosome of a pair contains a segment not present in its mate. The condition is found in about half the females from a stock culture of *Sciara ocellaris* Comstock, and may be concerned with genetic peculiarities in this species. It represents a normal condition, in the sense that no irradiation or other artificial treatment has been employed. The extra unit, shown in figure 1, is absent in other females of the culture. Preliminary observations on males indicate that both conditions are present also in this sex. The possible relation of the conditions to sex determination will be discussed elsewhere.

Attention will be confined here to two main problems dealing respectively with the longitudinal and the transverse differentiation of the chro-

mosomes in our material. The salivary gland chromosome is clearly differentiated longitudinally by the cross striations, which presumably reflect some kind of segmental organization. Similarly, in a transverse direction the chromosome (except the unpaired *X*) is compound at least to the extent of being double, for it arises by lateral fusion of two homologues. If the hypothesis of Koltzoff and Bridges is correct it is, in late stages, not only double, but composed of 8, 16 or 32 "gene strings" arranged in rosette fashion about the axis and at the same time spirally coiled.



FIGURES 1 AND 2

1, one end of a chromosome showing a half-segment; i.e., segment present in one component and not in the other. 2, the other end of this chromosome in two different specimens—*a*, prepared to show vesicular segments, *b*, prepared to show bands or discs. Camera drawings, by Dr. Helen B. Smith; all from *S. ocellaris* Comstock.

As noted in the other paper just mentioned, striking differences are found in the appearance of our material following different modes of dissection in preparation for fixing in the usual aceto-carmine mixture. After dissection in Ringer's solution, the banded structure tends to be prominent (Fig. 2*b*), while after dissection in the body fluid of the organism, or directly in the aceto-carmine fixative, biscuit-like or wafer-like segments appear prominently. In extreme cases the chromosomes are

full of these segments and few bands are to be seen (Fig. 2*a*). The biscuit-like structures are vacuolated in appearance and resemble compressed vesicles. For convenience they will be referred to here as "vesicles." They apparently arise from pre-existing segments of differentiated material in the chromosome, for they exhibit characteristic differences in size, shape and structural make-up, and exhibit a relatively constant pattern which corresponds in homologous chromosomes of different cells and individuals.

Study of living material has not yet sufficed to determine which of these conditions—that with discs or that with vesicles—more nearly approximates the normal. It suggests, however, that vesicles are present which have more nearly parallel faces than those found in the fixed material, and that the latter are somewhat swollen by the fixative.

In fixed preparations of the vesiculated type the space between successive vesicles or segments in the series is filled with material apparently different from that within the vesicles. The faces of the vesicles here (extending transversely across the chromosome) appear to represent boundaries, or interfaces, separating one kind of material from the other. By comparing corresponding series of bands (discs) and of vesicles in homologous chromosomes the relationship between the two types of conditions may be observed, as indicated in figures 2*a* and 2*b*. Certain discs or bands represent individual vesicles, while in other cases a pair of discs together with the intervening material combine to make a vesicle. In the latter case the two discs apparently represent the two faces of the vesicle.

In some cases a thick disc which represents a vesicle looks like a pair of discs closely applied to one another. This suggests that the thinner discs which individually form vesicles may have a similar structure. If this is true the structures which go to form vesicles have a similarity in organization, whether they appear as single bands or pairs of bands in preparations showing the banded structure.

Study of the internal structures of the discs or pairs of discs (in fixed material) indicates that definite differences in protoplasmic structure are associated with the different types of discs, suggesting chemical differences in the protoplasm. This feature will be considered more in detail elsewhere.

In addition to the discs which go to make vesicles there are numerous very thin ones, often appearing in the form of granular lines, which do not become vesiculated even in preparations showing the most extreme vesiculation. Some of these are in pairs and may represent segments which do not become vesiculated. Others appear to be single. Certain bands or lines are very faint, and seem not to represent differentiated structures, but merely to represent lines or rows of fine granules marking the boundary between what appear to be rows of alveoli in the protoplasm. If this latter inference is correct the more delicate "bands" may not be comparable to the more prominent ones.

Careful study of one region of one particular chromosome, using material prepared in various ways, has provided a definite pattern which includes all the segments, large or small, that appear regularly. A rough calculation, based on this pattern, gives the total number of segments in the haploid set of chromosomes as at least 1500 to 2000. Whether or not these segments represent individual gene loci is not yet clear, but their number is probably sufficient to meet known requirements.

In any event, it seems clear that the vesicles or segments, as defined above, represent definite units of structure in the chromosome. A striking illustration of this point is furnished by the special condition mentioned above and shown in figure 1. In one chromosome (which, of course, is double) one segment is represented by a vesicle that extends only half-way across. This vesicle is found regularly in each nucleus of these individuals, and is constant in form and position throughout. The evidence shows clearly that one of the two chromosomes of the pair possesses the segment and the other lacks it. The suggestion arises immediately that this may be a single gene locus.

The half-segment just described is of interest also in connection with the transverse organization of the chromosome. In the region carrying this half-segment, the two chromosomes of the pair are clearly not fused to form a cylinder, but are attached side by side like the two barrels of a double barreled shotgun.[6] Most of the segments are continuous, showing that fusion of the matrix at least, has taken place; but the outline of each original chromosome is distinct. It is clear that the half-segment lies in one of these chromosomes and not in the other or in an intermediate position.

Careful study of both fixed and living material indicates that this double type of structure is probably characteristic of all the chromosomes, in *Sciara*, and makes it seem probable that the two component chromosomes in each case are simply united side by side and, in this sense, retain their identity, instead of being fused into a cylinder with the components spirally coiled. If the segments are compound, therefore, through repeated division of the original chromioles, the series should, on this interpretation, extend in straight lines along the chromosome, or each component chromosome should have its own spiral organization. It is difficult to see how either of these conditions could exist, however, if each segment consists of a rosette of units and if a real fusion of the two original chromosomes has occurred. It may be noted also that if, as Painter (loc. cit.) has intimated in the case of *Drosophila*, the two homologous chromosomes unite in the salivary glands at a late stage, after they become large and the banded structure is well developed, it is difficult to see how any uniform, compound spiral organization could be developed in the structure as a whole. Presumably each component chromosome would have its own spiral organization by this time.

Conclusions on these points would be premature at present, but it may be noted that our evidence suggests a pairing of the two original chromosomes at a time when they are in a thread-like condition comparable to that of the zygotene in maturation, with intimate union chromiole for chromiole, and a subsequent retention of that relationship without sprial coiling. It further suggests that the "vesicles" or segments, of the present account, represent in some manner the chromioles considered by Belling[7] to represent genes. Whether each vesicle should on this view, represent two homologous chromioles partially or completely fused and greatly enlarged by growth, or each should be considered a compound of eight, sixteen, etc., units and the large size be considered a result of repeated division, is not clear. In our material, as already noted, each chromosome commonly appears as a ribbon-like structure, with a groove down the middle between the two components. This groove marks a definite constriction in the vesicles or discs, suggesting incomplete fusion. Indeed, the evidence may be held to suggest that the original chromioles are only closely appressed due to fusion of the surrounding matrix material of the two original chromosomes.

[1] This investigation has been aided by a grant from the NATIONAL RESEARCH COUNCIL, Committee for Research in Problems of Sex.

[2] *Zeits. Zellf. Mik. Anat.*, **17**, 67 (1933).

[3] *Science*, **78**, 585 (1933); *Genetics*, **19**, 175–448 (1934).

[4] Lecture at Carnegie Institution, Cold Spring Harbor, N. Y., Sept. 4, 1934.

[5] *Science*, **80**, 312 (1934).

[6] In many regions the chromosomes even appear thin and flat, like a lamp-wick.

[7] *Univ. of Calif. Pub. in Bot.*, **14**, 307 (1928).

BODY SIZE OF RECIPROCAL HYBRIDS IN RABBIT CROSSES

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In a previous publication[1] I have called attention to the theoretical importance of determining whether reciprocal crosses between races of animals of unlike body size give identical results.

The argument is as follows. If, as is assumed in the multiple factor theory of size inheritance, chromosomal genes alone function in the genetic determination of body size, then reciprocal crosses between pure races should give identical results, since the chromosome equipment of gametes is the same in eggs and in sperm. Strictly, this conclusion should be limited

to female offspring, since they alone (in mammals, as in *Drosophila*) would receive the same complement of sex chromosomes from each parent, irrespective of the nature of the cross, whereas male offspring would receive an *X*-chromosome from the mother and a *Y*-chromosome from the father, and the racial source of these would differ in reciprocal crosses. But as regards the autosomes, the chromosome equipment derived from each parent should be the same in offspring of both sexes.

For many years I have sought critical evidence as to whether genes located in particular chromosomes are concerned in the determination of body size. The results thus far obtained have been entirely negative in crosses between large-bodied and small-bodied rabbits. But Green[2] reports genetic linkage between large size of body and brown coat color in a species cross of mice. The qualitatively different result in the two cases requires explanation and is being given further investigation.

In 1929[3] I reported the results of a cross between a race of small rabbits averaging less than 1500 grams in adult body weight and a race of large rabbits averaging over 5000 grams in adult weight. The cross was made chiefly between large race females and small race males, but a few litters were produced from the reciprocal cross in which the mothers were of small race. In table 1 and figure 2 of that paper data are given on the weight of the reciprocal hybrids from birth to age 52 days. These observations were made on a group of 18 F_1 young borne by large mothers and a group of 8 young borne by small mothers. The latter are consistently smaller throughout the entire period, but the difference is not great. At birth the 8 young by small mothers averaged 55.3 grams; the 18 young by large mothers averaged 59.4 grams, being some 7 per cent heavier. At 52 days of age the respective average weights of the two groups were 1019 and 1055, the latter being about 3.5 per cent heavier. This small difference was considered negligible for the object then in mind—possible genetic linkage between four independent color genes and body size.

In view of the negative result of that investigation, it has seemed desirable to look for a genetic agency other than chromosomes as a possible explanation of differences in the results of reciprocal crosses. But first it should be established beyond question that such differences exist. Consequently a renewed investigation has been made of reciprocal crosses between races of large-bodied and small-bodied rabbits.

As a large race, an inbred strain of New Zealand White rabbits has been used which ranges in adult weight from 3500 to 4000 grams. Crosses with this race have been made reciprocally, using three different small races, which we may call *A*, *B* and *C*. *A* is identical with the small race of my 1929 paper and is under 1500 grams in average adult weight. *B* is derived from *A*, in large part, but is perhaps 50 grams heavier, on the average.

C is wholly unrelated to *A* and *B*, consisting of a standard bred race of Himalayans weighing about 2000–2500 grams.

The young produced by these reciprocal crosses were kept under optimum conditions for growth until they were four months old. The majority of them were nursed by foster mothers, and not more than four young were nursed by the same female, in order that each one might have abundant food. The young were weighed weekly from the time of weaning (about 30 days of age) until age 120 days. Although they would not be full grown at this time, their relative sizes should be clearly established without complications due to puberty changes. Female rabbits are slightly heavier than males when full grown, owing chiefly to the earlier arrest of growth in males by the approach of sexual maturity. But at four months of age this difference has scarcely become established, and so both sexes can be treated together statistically. Nevertheless the data for each sex have been presented separately as well as combined in table 1.

TABLE 1

COMPARATIVE WEIGHTS AT FOUR MONTHS OF AGE OF RECIPROCAL HYBRIDS BETWEEN LARGE RACE AND SMALL RACE RABBITS

	NO.	AV. WT.		NO.	AV. WT.
♀N.Z.W. × ♂ Race *A*			♀ Race *A* × ♂ N.Z.W.		
Males	5	1964	Males	4	1825
Females	7	1978	Females	6	2043
Both sexes	12	1972	Both sexes	10	1956
♀N.Z.W. × ♂ Race *B*			♀ Race *B* × ♂ N.Z.W.		
Males	10	2148	Males	49	1922
Females	7	1978	Females	52	1913
Both sexes	17	2078	Both sexes	101	1917
♀N.Z.W. × Race *C*			♀Race *C* × ♂ N.Z.W.		
Males	4	2027	Males	7	2042
Females	6	2130	Females	17	2094
Both sexes	10	2089	Both sexes	24	2070
All three crosses, males	19	2074	All three crosses, males	60	1930
All three crosses, females	20	2024	All three crosses, females	75	1965
All three crosses, both sexes	39	2048	All three crosses, both sexes	135	1949

A litter of six pure bred New Zealand White rabbits averaged 2432 grams in weight at age 120 days, whereas a litter of four Race *B* animals at the same age averaged 1215 grams. The hybrids between the two races show the characteristic expression of hybrid vigor, accelerated early growth, which makes them at four months old much nearer the large parent race. See table 1, second section.

The average weight of each category of young produced by New Zealand White mothers, when mated with a small race male, is shown in the left half of table 1. The corresponding data for the reciprocal cross are

shown in the right half of table 1. In four out of six cases, the young of a New Zealand White mother exceed in weight the young of the same sex produced in the reciprocal cross. The two cases in which this is not true are the female offspring of the Race *A* cross and the male offspring of the Race *C* cross, in which cases averages are based on very small numbers of individuals (4–6). Combining the data for males and females, we find that in all three crosses the average weight of the young produced by a New Zealand White mother exceeds that of the young produced by the reciprocal cross.

If we combine the data for all three crosses (last row of table 1), it is found that the 39 young of New Zealand White mothers mated with small race males have an average weight at four months old of 2048 grams, whereas the 135 young of small mothers mated with New Zealand White males have an average weight of 1949 grams. The difference is 99 ± 19.7 grams, or five times the probable error of the difference, which is clearly significant.

The variation of each of the two groups of young considered separately is shown in table 2. It is clear from an examination of these tables that

TABLE 2

VARIATION IN WEIGHT AT FOUR MONTHS OF THE YOUNG PRODUCED BY THE RECIPROCAL CROSSES

	WEIGHT CLASSES IN HECTOGRAMS										NO.	MEAN	S. D.
	15	16	17	18	19	20	21	22	23	24			
♀♀N.Z.W. × ♂♂ Races A–C		1	1	1	13	11	4	5	2	1	39	2048 ± 17	160 ± 12
♀♀Races A–C × ♂♂N.Z.W.	5	6	16	22	26	34	16	8	1	1	135	1949 ± 10	172 ± 10

New Zealand White females, when mated with small race males, produce offspring some five per cent heavier at four months of age than young produced by the reciprocal cross. This is in agreement with the observations reported in 1929 on a smaller number of young but from reciprocal crosses between races still more widely separated in body size. We may conclude, on the basis of both experiments, that something besides chromosomal genes is concerned in the determination of body size in crosses between races of unlike size. Conceivably it may be either genetic or non-genetic. If genetic, the egg cytoplasm would seem to be the most likely agency, as suggested in my 1933 paper. If non-genetic, some agency supplied by the mother during gestation might be held responsible for the observed difference. A decision between these alternative explanations cannot be made at present, but must await further experimental study of the case.

[1] "The Gene Theory in Relation to Blending Inheritance," *Proc. Nat. Acad. Sci.*, **19**, 1011–1015 (1933).

[2] Green, C. V., "Further Evidence of Linkage in Size Inheritance," *Amer. Nat.*, **67**, 377–380 (1933).

[3] "A Further Study of Size Inheritance in Rabbits, with Special Reference to the Existence of Genes for Size Characters," *Jour. Exp. Zoöl.*, **53**, 421–454 (1929).

A HYPERTRAGULID FROM THE SESPE UPPERMOST EOCENE, CALIFORNIA

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Introduction.—It becomes increasingly clear that the faunal stage represented at Locality 150 in the Sespe deposits north of the Simi Valley, California, marks an advance beyond the stage or stages recorded lower in the stratigraphic sequence of the Sespe as exposed in this region. Determination of age of this fauna as Eocene is maintained for the reasons stated in the first paper[1] announcing the discovery of Eocene mammals in California. Future survey of the fauna as a whole and recognition of relationships of the assemblage to comparable faunas found elsewhere may contribute toward elevating rather than toward depressing the fauna in the geologic time scale. It suffices for the present to state that the assemblage appears to be close to or within the Eocene-Oligocene transition period in the Tertiary history of vertebrate life for North America. In order to distinguish this stage from the upper Eocene fauna or faunas occurring lower in the Sespe section, the age designation uppermost Eocene is now applied to the assemblage.

The following mammals from Locality 150 have been specifically determined:

Hyænodon vetus
Hyænodon (Protohyænodon) exiguus
Pterodon californicus
Pleurocyon (Simidectes) merriami
Chumashius balchi
Amynodontopsis bodei

To this list is now added a hypertragulid related to the Oligocene genus *Hypertragulus*.

FAMILY HYPERTRAGULIDAE

Simimeryx hudsoni, n. gen. and n. sp.

Type Specimen.—Fragment of palate with $P\underline{2} - M\underline{3}$, No. 1764 C.I.T. Vert. Pale. Coll., plate 1, figures 1, 1*a*.

Paratype.—Ramus of mandible with $P\overline{1} - M\overline{3}$, No. 1244 C.I.T., plate 1, figures 2, 2*a*.

Referred Specimen.—A fragment of the maxillary with $P\underline{3}$ and $P\underline{4}$, No. 1354, plate 1, figure 3.

Locality.—Sespe deposits, north of the Simi Valley, Ventura County, California, Locality 150 C.I.T. Vert. Pale.

Generic and Specific Characters.—Molar teeth with shorter crowns than in *Hypertragulus*. Upper molars with anterior intermediate cusp (protoconule); parastyle prominent and broadly rounded, not compressed anteroposteriorly as in *Hypertragulus*. Mesostyle absent, but external cingulum present. Internal cingulum present at base of posterior crescent in $M\underline{1}$ and $M\underline{2}$. Shelf present on inner side of molar teeth between inner crescents, but no pillar. Inner cusp on $P4$ not crescentic. $P\overline{2} - P\overline{4}$ form a closed series. $P\overline{4}$ with shallow posterior basin. Size near that of *Hypertragulus calcaratus* or smaller. The species is named for Dr. Frank S. Hudson of the Shell Company of California, who directed my attention to the Sespe deposits as exposed north of the Simi Valley.

Comparisons.—Remnants of the palate in the type specimen show the position of a palatine foramen opposite the posterior half of $P\underline{3}$. Three upper premolars are present in the type specimen, although $P\underline{3}$ is imperfectly preserved. The premolars are distinctly more like those of *Hypertragulus* than like those of *Leptomeryx*. $P\underline{2}$ is reduced in size and, as in *Hypertragulus*, has a simple laterally compressed crown. $P\underline{3}$ is complete in the referred specimen, No. 1354. The external surface of this tooth does not show quite so well marked a groove or concavity posterior to the principal cusp. However, the concavity is much better defined in the type specimen, No. 1764, than in No. 1354. The tooth is wider transversely and the inner cuspule is more distinct than in *Hypertragulus*. Moreover, the anterior basal tubercle is better developed than in the latter. In $P\underline{4}$ the posterior cingulum extends well around on the inner side of the base of the inner cusp and may actually encircle the inner side as in the referred specimen, No. 1354. Although the posterior wing of the inner cusp may be but feebly developed in *Hypertragulus*, it is absent in *Simimeryx* and there is a distinctly less tendency in the latter to outline a fossette between the opposed walls of the outer and inner cusps than in the Oligocene genus.

In each of the upper molars, the transverse diameter measured across the protocone is distinctly greater than that across the posterior crescent, and this feature becomes progressively more marked from the first to last molar, inclusive. The crowns of these teeth are strikingly like those in *Hypertragulus* with some noteworthy exceptions. Distinctly primitive features, in which the molars differ from those of the Oligocene genus, are (1) shorter crowns and (2) presence of a protoconule. This tiny but well marked cuspule is present on each of the molars in the type specimen. As

in *Hypertragulus*, a mesostyle is absent. In this respect *Simimeryx* differs from *Leptomeryx* and from *Bunomeryx*. The parastyle forms a broadly rounded, antero-external pillar, which differs noticeably in shape from the compressed style seen in *Hypertragulus*. The parastyle connects with the anterior crest extending outward from the antero-internal cusp as in the Mongolian genus *Archæomeryx*.

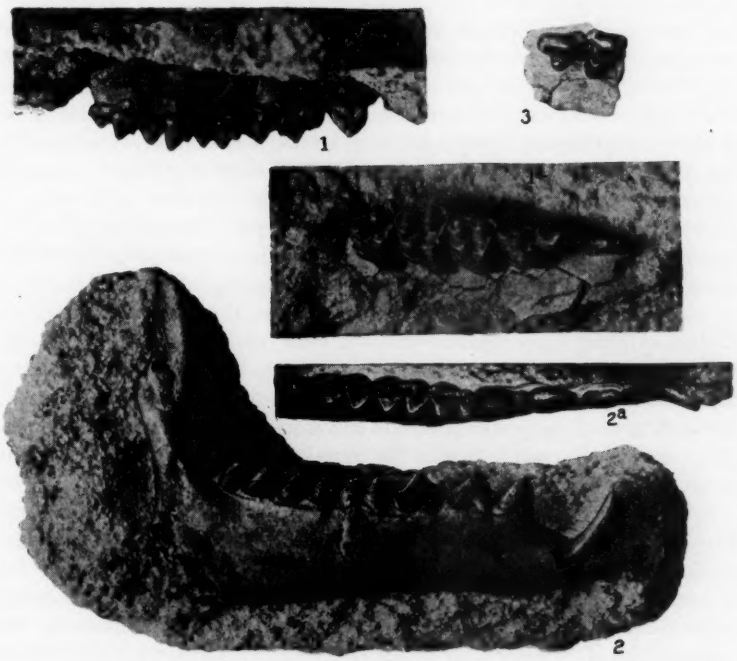

PLATE 1

Simimeryx hudsoni, n. gen. and n. sp.

Figures 1, 1*a*, type specimen, fragment of maxillary with $P\underline{2} - M\underline{3}$, No. 1764; lateral and occlusal views; × 1½.

Figures 2, 2*a*, paratype, right ramus with $P\overline{1} - M\overline{3}$, No. 1244; lateral and occlusal views; × 1½.

Figure 3, referred specimen, fragment of maxillary with $P\underline{3}$ and $P\underline{4}$, No. 1354; occlusal view; × 1½.

California Institute of Technology Collections. Sespe Uppermost Eocene, California.

In the ramus of the mandible an anterior mental foramen is situated beneath $P\overline{2}$ and a posterior foramen may be present below $P\overline{4}$. Unfortunately, with the exception of the lower caniniform tooth, nothing is known concerning the anterior teeth. As in *Hypertragulus*, a diastema separates

$P\overline{2}$ from a caniniform tooth in front of it, but this hiatus in the tooth-row is not so great as in the Oligocene form. The three posterior premolars form a closed series in *Simimeryx*, whereas in *Hypertragulus* $P\overline{2}$ is separated from $P\overline{3}$ by a short diastema. One may infer, perhaps, from the lack of spacing of the second lower premolar that the snout in the Sespe form was not so long as in *Hypertragulus*. $P\overline{2}$, as viewed from the external side, resembles that tooth in *Hypertragulus*. In $P\overline{4}$ the anterior crest extending downward and forward from the principal cusp turns inward and terminates in an antero-internal pillar. On the specimens available showing this tooth no separation is seen between the principal cusp and an internal accessory cusp. These two cusps are closely joined, if indeed the latter is present at all. Only a slight separation is evident in *Hypertragulus*, but in *Leptotragulus* a distinctly formed inner cusp is present. In *Simimeryx* the posterior portion of the crown forms a shallow basin which is bordered on three sides by crests. In *Leptotragulus* the rim bordering the basin may be more strongly developed and a minute spur projects into the basin. The external crescents of the lower molars exhibit less tendency to join with the inner cusps. Thus, for example, the posterior wing of the postero-external cusp remains distinct from the posterior ridge of the postero-internal cusp in moderately worn teeth, while in *Hypertragulus* a firm union has been established between the two at this stage. Similarly, the inner wall of the posterior lobe in $M\overline{3}$ is not joined firmly with the posterior crest of the cusp in front. No inner cusp is present on the inner rim of the posterior lobe in $M\overline{3}$ as in *Leptotragulus*.

Relationships.—The characters presented in the dentition of *Simimeryx* strongly suggest a position for this genus in or near the line of development leading toward *Hypertragulus*. The occurrence of the Sespe type in the sequence of Tertiary faunal stages accords with this view. Although relationship to the Oligocene and lower Miocene genus is apparent, the relationships of *Simimeryx* to earlier Eocene artiodactyls are more difficult to establish on the basis of present information.

The upper molars in the Uinta genus *Bunomeryx* possess a mesostyle. Among the several genera described by Peterson[2] from the Uinta, *Hylomeryx* does not appear to have any special relationship to our type. *Sphenomeryx* resembles *Simimeryx* in absence of mesostyle in upper molars but differs in presence of a reduced parastyle in these teeth, as well as in the farther posterior position of the inner cusp in $P\underline{3}$. *Mesomeryx*, on the other hand, appears to make a closer approach to the Sespe genus. However, the following differences are noted when *Simimeryx* is compared with this form: (1) in $P\underline{3}$ the inner cusp is farther removed from the principal cusp and is much better defined. As a result the basal outline of the tooth is wholly different. Moreover, the principal cusp is farther removed from the anterior end of the tooth. (2) A greater discrepancy in size exists be-

tween the inner and outer cusps in $P\underline{4}$; (3) The molars have a different shape, being longer anteroposteriorly with reference to their width. Furthermore, the inner anterior corner in these teeth is more prominent. The molars in the Uinta genus are described as being rather more selenodont than bunodont. In *Simimeryx* the anterior crest of the anterointernal cusp extends toward and is connected with the parastyle, whereas in $M\underline{1}$ and $M\underline{2}$ of *Mesomeryx* such a connection is not established. Although a protoconule has not been recognized in the type specimen of *Mesomeryx*, it is possible that this cuspule has been obliterated through wear.

Archæomeryx of the later Eocene Shara Murun formation, Mongolia,[3] resembles the Sespe type rather closely in stage of evolution of the cheek-teeth. Significant differences in the upper molar dentition are (1) presence of a mesostyle and (2) absence of the anterior intermediate cusp (protoconule). Less resemblance to the Californian form is displayed by the species of *Lophiomeryx* and *Miomeryx*, described from the Ardyn Obo of Mongolia.[4]

MEASUREMENTS (IN MILLIMETERS)

	Simimeryx hudsoni, n. gen. and n. sp. Type No. 1764 C.I.T.
Length from anterior end of $P\underline{2}$ to posterior end of $M\underline{3}$	28.7
Length from anterior end of $P\underline{2}$ to posterior end of $P\underline{4}$	13.6
Length of molar series from antero-external corner of $M\underline{1}$ to postero-external corner of $M\underline{3}$	15.7
	Simimeryx hudsoni, n. gen. and n. sp. Paratype No. 1244 C.I.T.
Length from anterior end of $P\overline{1}$ to posterior end of $M\overline{3}$	42.4
Length from anterior end of $P\overline{2}$ to posterior end of $M\overline{3}$	33.8
Length of diastema in front of $P\overline{2}$	4.9
Length from anterior end of $P\overline{1}$ to posterior end of $P\overline{4}$	22.9
Length of molar series	19.5
Depth of ramus below anterior end of $M\overline{3}$	10.3
Depth of ramus below anterior end of $P\overline{2}$	7.3

[1] Stock, C., *Proc. Nat. Acad. Sci.*, **18**, 523 (1932).

[2] Peterson, O. A., *Ann. Carnegie Mus.*, **12**, 66–75, pls. 36–37 (1919).

[3] Matthew, W. D., and Granger, W., *Amer. Mus. Nov.*, No. 196 (1925).

[4] Matthew, W. D., and Granger, W., *Ibid.*, No. 195 (1925).

ULTRA-VIOLET TRANSMISSION CHANGES IN GLASS AS A FUNCTION OF THE WAVE-LENGTH OF THE RADIATION STIMULUS

By W. W. Coblentz and R. Stair

National Bureau of Standards

Communicated November 15, 1934

During the past two years the writers have been investigating photochemical changes in glass produced by exposure to homogeneous ultra-violet radiation of different wave-lengths of known energy value. A description of the preliminary results obtained is in press.[1] Recently we have extended the calculations of these data, the results of which appear worthy of permanent record.

The experimental procedure of this investigation is to expose the glass (e.g., a soda-lime-silica glass, highly transparent to short wave-length ultra-violet radiation) to spectroscopically isolated, homogeneous, ultra-violet radiation, from a quartz mercury arc lamp, which has the property of profoundly modifying the ultra-violet transparency of the glass.

This change in spectral transmission (which differs in magnitude for exposure to different wave-lengths), occurs in a wide band, that begins at about 405 mμ and extends, without indentations, to an undetermined wave-length into the extreme ultra-violet, where it is masked by the great opacity characteristic of glasses containing silica.

In the absence of a chemical analysis, the absorption spectrum is used as a criterion for judging the magnitude of the photochemical change in the glass.

Since the ultra-violet spectral transmission curves, resulting from exposure to radiation of different wave-lengths and with time of exposure to a given wave-length, are similar, it is sufficient to determine the change in transmission by making observations at wave-length 302 mμ.

The outstanding results of this research are that in the soda-lime-silica and soda-silica[2] glasses examined (whether or not they contained an appreciable amount of iron oxide), stages in photochemical equilibrium were found, which are different for each wave-length of homogeneous radiation to which the glass reacts. In contrast, a sample of potash-lime-silica glass, which in a thickness of 2.6 mm. is highly transparent to the ultra-violet (transmission = 16% at 254 mμ) was not appreciably affected by ultra-violet radiation—other than a slight blackening at the surface exposed to the lamp. From the data on these three kinds of glass it appears that the soda is the photosensitive constituent in a soda-lime-silica glass.

At each equilibrium level the soda-lime-silica glass seems to behave like any non-photosensitive glass. To attain this equilibrium, the reaction,

caused by a given wave-length, is in the direction to either increase or decrease the transmission, depending upon the existing condition developed by previous heat treatment or by previous exposure to other wave-lengths to which the glass reacts.

Using homogeneous radiation, no wave-length (at least not for wave-lengths shorter than and including 365 mμ) was found that has the exclusive property of either increasing or decreasing the ultra-violet transmission in the soda-lime-silica glasses examined.

The longest wave-length having an appreciable photochemical action on a soda-lime-silica glass is at about 405 mμ. For wave-lengths 365 mμ and shorter, the photochemical action, as determined by the equilibrium levels in spectral absorption, is a function of the wave-length (the frequency)

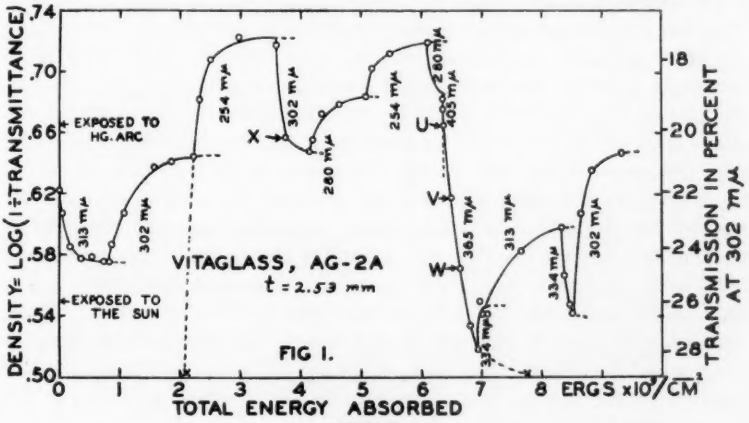

FIGURE 1

of the radiation stimulus; the shorter the wave-length the greater the photochemical action in the direction of greater absorption.

In contrast with a soda-lime-silica glass, a phosphate-lime glass (containing oxides of Al, B, Na, Mg, Si and Fe as minor constituents) is depreciated in ultra-violet transmission only by wave-lengths shorter than about 290 mμ. After depreciation, the ultra-violet spectral transmission curve of the phosphate-lime glass is similar to that of a soda-lime-silica glass. Nevertheless, exposure of the depreciated phosphate-lime glass to homogeneous radiation of wave-lengths 297 to 365 mμ does not appreciate the transmission, as is the case in the soda-lime-silica glass. In other words, in contrast with the soda-lime-silica glass, certain wave-lengths have the property of decreasing the ultra-violet transmission of a (Corex A) phosphate-lime glass; but in the spectral range investigated no reverse reaction was observed.

In figure 1 are shown the observed changes in density (transmission)

produced by different wave-lengths, depicted in the order in which the observations were made, beginning with homogeneous radiation of wave-length at 313 mμ as the stimulus. Prior to this the glass had been exposed to the total radiation from a quartz mercury arc which reduced the transmission to the point indicated on the chart. The abscissas represent the total radiant energy absorbed in the glass, no distinction being made between the amount absorbed by the main constituents of the glass, and the amount absorbed by the photosensitive substance which undergoes a photochemical reaction, indicated by the change in transmission. The ordinates indicate the optical density [= log (1 ÷ transmittance)] of the glass at wave-length 302 mμ. For convenience in reference the scale of the per cent transmissions is also indicated.

The curves in figure 1 show the energy absorbed in the glass in changing from one equilibrium level to another, and serve the purpose of depicting the observations in a connected manner; but without further data, they cannot be used to elucidate several important questions. For example, the rate of transition in going from one equilibrium level to another depends upon the starting point and cannot be determined directly from the data illustrated in figure 1. Thus, the slopes of the transmission curves for irradiation with the 365 mμ-line and with the 254 mμ-line are not comparable, because the one was observed at the initial stage and the other at the final stage of the reaction. The slope of the recovery curve using the 365 mμ-line appears steeper than the depreciation curve using the 254 mμ-line. However, if the latter curve had been observed beginning at the equilibrium level at 29 per cent, as indicated by the dotted line in figure 1, the slope of the transmission curve at the initial part of the photochemical reaction to the 254 mμ-line appears to be just as steep as that for the 365 mμ-emission line at its initial stage.

Manifestly, as indicated in figure 1, the total energy required to attain equilibrium depends upon the existing condition of the photosensitive material. From an inspection of the sections of the curves in figure 1, it appears that the total energy absorbed in going from one equilibrium level to a lower equilibrium level is greater than that absorbed in returning to the original equilibrium level. Further investigation is required to determine whether this is generally true, and whether it is owing to a difference in the heat of formation and heat of dissociation of the photosensitive substance.

As already noted the main constituents of the glass are almost completely opaque to radiation of wave-lengths 280 mμ and shorter, whereas at wave-lengths longer than 365 mμ (to 405 mμ) only 2 to 3 per cent of the incident radiation is absorbed by the glass. From this it can be seen that, for wave-lengths 254 mμ and shorter, the depth of penetration is shallow and that the greatest photochemical action occurs near the surface of the glass fac-

ing the incident radiation. On the other hand, for the wave-lengths 365 mμ and longer, the penetration is deep and, during the process of formation and disintegration of the photosensitive substances, the concentration is more uniform throughout the irradiated layer of glass.

In view of the foregoing considerations it is evidently desirable to obtain some idea of the energy relations directly involved in the photochemical reactions. Without a chemical analysis data are lacking regarding the concentration of the photosensitive substance. The only basic reference point is an equilibrium state where (as evidenced by the constant value of

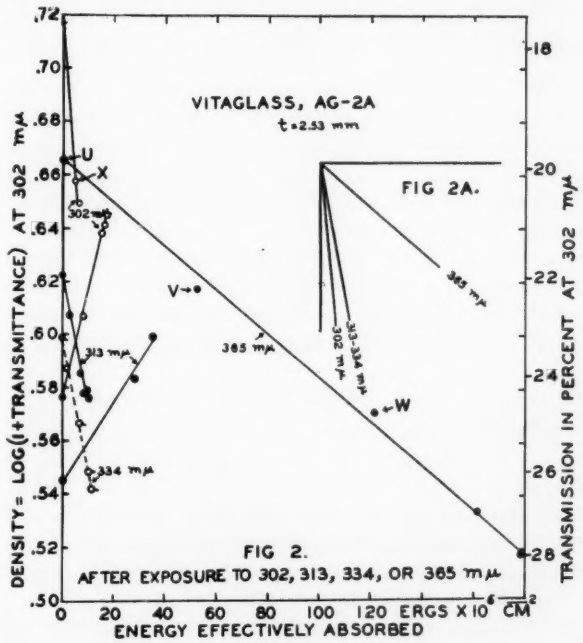

FIG. 2.
AFTER EXPOSURE TO 302, 313, 334, OR 365 mμ

FIGURE 2

the transmission) no further chemical change is produced by a given wave-length, and the energy absorbed is apparently converted into heat, just as presumably occurs in any glass not containing a photosensitive material. Using these equilibrium levels as reference points, and using the observed changes in transmission (density) as a measure of changes in concentration of the photosensitive substance, the energies involved in the formation and disintegration of the photosensitive substance are worked out in the following manner. At any given absorption (transmission) level above or below the equilibrium state it is assumed that the absorbed energy, which is used

in producing a chemical change, is proportional to the difference between the per cent absorption at the given level and at the equilibrium level. For an interval in which there is a continuous change in absorption the total chemical change is assumed to be proportional to the average excess (or shortage) of the absorption at the given interval over that at the equilibrium state. For the small concentrations of the substances formed (or decomposed), the average total energy utilized during a given interval should be independent of whether the maximum absorption is at the beginning or the end of the exposure.

The following numerical examples are given showing how the various points of the curves in figure 2 are obtained. The wave-length of exciting radiation at 365 mμ (points U, V and W in Fig. 1) is selected, since it requires auxiliary data, on the per cent transmission (absorption) at this wave-length, not shown in the illustration. For the first exposure interval (U to V in Fig. 1) the observed initial and final transmissions at 302 mμ were 20.0% and 22.4%, respectively. The corresponding absorption values at 365 mμ are 11.8% and 9.7% and the equilibrium level of absorption at 365 mμ corresponding to a transmission of 29% at 302 mμ, is 5.2%. The change from 11.8% to 9.7% gives an average difference of 5.55% above the equilibrium level (5.2%). The product of this difference (5.55%) the intensity (1105 μw/cm.^{2}) and the time (23.75 hours) gives the energy (52.4 $\times$ 10^6 ergs/cm^2.)/associated with this interval—point V, figure 2.

For the point W, the transmission values at 302 mμ are 22.4% and 24.9%, respectively. The corresponding absorption values at 365 mμ are 9.7% and 7.7%. This gives an average difference of 3.5% above the equilibrium level (5.2%). The product of this difference (3.5%), the intensity (1105 μw/cm.^{2}) and the time (49.6 hours) gives the energy (69.1 $\times$ 10^6 ergs/cm.^{2}) associated with this interval. Adding this to the value (52.4 $\times$ 10^6 ergs/cm.^{2}) for the previous interval gives a total of 121.5 $\times$ 10^6 ergs/cm.^{2}—point W in figure 2.

Using the emission line at 302 mμ for the exciting radiation the calculations are more obvious. For example in the first test of appreciation in transmission of the glass by exposure to the emission line at 302 mμ (point X, Fig. 1), the transmission increased from 17.7 per cent to 20.3 per cent, (the absorption decreased from 78.1 to 75.5%)—an average difference of 1.9 per cent below the equilibrium level (20.9 per cent transmission; 74.9 per cent absorption) during this interval. For a linear change in absorption, the product of this difference (1.9), the intensity (405 μw/cm.^{2}) and the time of exposure (14.7 hours) gives the energy (4.1 $\times$ 10^6 ergs / cm.^{2}), associated with the change noted during this interval—see X in figure 2.

The data at hand are not sufficient for a complete analysis, for all the wave-lengths used. From the slopes of the lines, in figure 2 A, it appears

that the longer the wave-length (the lower the frequency) of the exciting radiation, the greater the amount of energy associated with the same photochemical change in the glass. The straight line relation of the data, indicated in figure 2, is suggestive and requires further investigation. This method of portraying the observations is not entirely free from shortcomings. However, if further research substantiates the correctness of the procedure it should prove a useful method of analyzing this type of photochemical reaction.

As a final remark it is relevant to note that, in the spectral range investigated, the shortest and the longest wave-lengths have the power of producing the greatest photochemical changes; the former in depreciating, the latter in appreciating the ultra-violet transmission of a soda-lime-silica glass.

[1] Coblentz, W. W., and Stair, R., *Jr. of Res., Nat. Bur. Stds.*, **13**, 773, Dec. (R.P. 744) (1934).

[2] The samples of soda-lime-silica, soda-lime and potash-lime-silica glass, composed of pure materials, melted in a platinum crucible, were supplied by A. N. Finn of the Glass Section of the National Bureau of Standards.

ON THE EXPANSION OR CONTRACTION OF A SYMMETRICAL CLOUD UNDER THE INFLUENCE OF GRAVITY

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Communicated October 27, 1934

1. Tolman[1] has undertaken a direct investigation of the history of a universe possessing radial symmetry, in which the matter consists of particles whose world-lines form a congruence (i.e., there is no chaotic motion). In the present note I do not pursue the method of Tolman, but suggest another point of view, which has much to recommend it on the score of mathematical simplicity. The starting point of investigations of this sort is usually the set of field equations[2]

$$R_{ij} + \lambda g_{ij} = -\kappa(T_{ij} - \tfrac{1}{2}g_{ij}T). \qquad (1)$$

Their use in this form involves the calculation of the Ricci tensor R_{ij}. In the present paper no such calculation is necessary: we employ invariant equations consequent on (1).

2. Let there be a universe containing ordered matter, that is, a system of particles without chaotic motion and without interaction except through the field equations (1). The congruence of their world-lines is specified

by the unit tangent vector μ^i: these world-lines are of course geodesics. Let the proper density be ρ. We shall assume that the universe is symmetric with respect to one of these geodesics C_0. We shall not trouble to give a formal definition of the word "symmetric" in this connection; the significance attached to it will be evident in the sequel. We know[3] that if ν^i is any unit vector, then

$$-\epsilon(\nu)R_{ij}\nu^i\nu^j = \Sigma K, \tag{2}$$

where $\epsilon(\nu)$ is the indicator of ν^i, chosen equal to $+1$ or -1 to make

$$\epsilon(\nu)\nu_i\nu^i = +1,$$

and ΣK is the sum of the Riemannian curvatures of the three 2-elements containing ν^i and another vector of an orthogonal tetrad having ν^i for a member.

Let us choose at any point on C_0 an orthogonal tetrad, one of whose constituent vectors (labeled 4) points along C_0; let the other three members be labeled 1, 2, 3. Let us denote the corresponding Riemannian curvatures by

$$K_{23}, K_{31}, K_{12}, K_{14}, K_{24}, K_{34}.$$

The assumed symmetry of space-time with respect to C_0 tells us that

$$K_{23} = K_{31} = K_{12} \ (= K', \text{ say}); \quad K_{14} = K_{24} = K_{34} \ (= K, \text{ say}). \tag{3}$$

Let us take for space-time the signature $+ + + -$, so that the indicator of μ^i is $\epsilon(\mu) = -1$.

Let us apply (2) at a point on C_0, taking first $\nu^i = \mu^i$. It gives

$$R_{ij}\mu^i\mu^j = K_{14} + K_{24} + K_{34} = 3K. \tag{4}$$

Now let ν^i be the unit vector 1 of the orthogonal tetrad: we get

$$-R_{ij}\nu^i\nu^j = K_{12} + K_{13} + K_{14} = K + 2K'. \tag{5}$$

But the field equations (1) give

$$\begin{aligned} R_{ij}\mu^i\mu^j - \lambda &= -\kappa(T_{ij}\mu^i\mu^j + \tfrac{1}{2}T), \\ R_{ij}\nu^i\nu^j + \lambda &= -\kappa(T_{ij}\nu^i\nu^j - \tfrac{1}{2}T), \end{aligned} \tag{6}$$

and so, with (4) and (5), we have

$$\left.\begin{aligned} K &= \tfrac{1}{3}\lambda - \tfrac{1}{3}\kappa(T_{ij}\mu^i\mu^j + \tfrac{1}{2}T), \\ K + 2K' &= \lambda + \kappa(T_{ij}\nu^i\nu^j - \tfrac{1}{2}T). \end{aligned}\right\} \tag{7}$$

For an ordered system of particles of the type considered

$$T_{ij} = \rho\mu_i\mu_j, \quad T = -\rho; \tag{8}$$

when these are substituted in (7) we get

$$\left.\begin{aligned} K &= \tfrac{1}{3}\lambda - \tfrac{1}{6}\kappa\rho, \\ K' &= \tfrac{1}{3}\lambda + \tfrac{1}{3}\kappa\rho. \end{aligned}\right\} \tag{9}$$

These are the two principal Riemannian curvatures at any point on the axis of symmetry of the universe.

3. Let us now consider the cloud of particles in the immediate neighborhood of C_0. Let η be the magnitude of the normal displacement from C_0 to a neighboring world-line C: it satisfies the differential equation[4]

$$\frac{d^2\eta}{ds^2} - K\eta = \eta \frac{\delta\theta_i}{\delta s}\frac{\delta\theta^i}{\delta s}, \tag{10}$$

where s is the arc of C_0, and θ^i is a unit vector in the direction of the displacement η. But it follows from the assumed symmetry of the distribution that θ^i is propagated parallelly along C_0: the right-hand side of (10) vanishes, and we have

$$\frac{d^2\eta}{ds^2} - K\eta = 0, \tag{11}$$

or, by (9),

$$\frac{d^2\eta}{ds^2} - (\tfrac{1}{3}\lambda - \tfrac{1}{6}\kappa\rho)\eta = 0. \tag{12}$$

But proper mass is conserved: this is an analytical consequence of the field equations (1) and the form (8) for the energy tensor. Therefore, again using the assumed symmetry,

$$\rho\eta^3 = \text{const.} = \rho_0\eta_0^3, \tag{13}$$

where ρ_0 is the proper density and η_0 the value of η for $s = 0$. Substituting in (12), and writing

$$\xi = \eta/\eta_0, \tag{14}$$

we have

$$\frac{d^2\xi}{ds^2} - \tfrac{1}{3}\lambda\xi + \tfrac{1}{6}\frac{\kappa\rho_0}{\xi^2} = 0. \tag{15}$$

This is the differential equation which controls the expansion or contraction of the small cloud of particles in the neighborhood of C_0, ξ being the ratio of the radius of the cloud at any instant to the initial radius.

4. Instead of integrating (15) formally, it is more interesting to interpret the results graphically. Let us define

$$v = d\xi/ds, \tag{16}$$

so that $\eta_0 v$ is the rate of increase of the radius of the cloud, measured with respect to proper time of C_0. Then

$$\frac{d^2\xi}{ds^2} = v\frac{dv}{d\xi}, \tag{17}$$

and (15) becomes

$$v\frac{dv}{d\xi} - \tfrac{1}{3}\lambda\xi + \tfrac{1}{6}\frac{\kappa\rho_0}{\xi^2} = 0, \tag{18}$$

or, on integration,

$$v^2 - \tfrac{1}{3}\lambda\xi^2 - \tfrac{1}{3}\frac{\kappa\rho_0}{\xi} = A, \tag{19}$$

a constant.

We now prepare a diagram (Fig. 1), in which (ξ, v) are represented by rectangular Cartesian coördinates. Each curve corresponds to an assigned value of the constant A, and represents a possible history of the cloud. The arrows indicate the sense in time: since $ds > 0$, it follows from (16) that

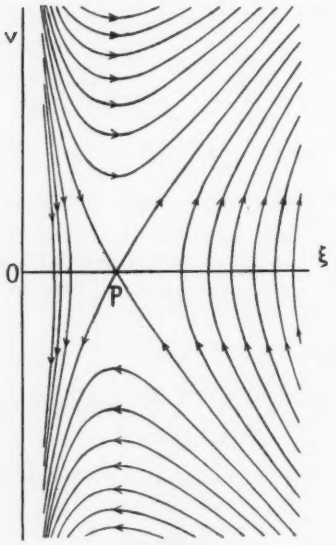

FIGURE 1

$$d\xi/v > 0, \tag{20}$$

so that ξ is increasing for positive values of v, and decreasing for negative values of v.

It follows from (18) that when $v = 0$ we have an infinite value for $dv/d\xi$, except when

$$\xi = \left(\frac{\kappa\rho_0}{2\lambda}\right)^{1/3}. \tag{21}$$

This gives a singular point P in the diagram. The value of A for the curves through this point is

$$A = -\tfrac{1}{2}\kappa\rho_0\left(\frac{2\lambda}{\kappa\rho_0}\right)^{1/3} = -\lambda\left(\frac{\kappa\rho_0}{2\lambda}\right)^{2/3}. \tag{22}$$

The following fact is immediately evident from figure 1: *No oscillation of the cloud is possible: its radius has at most one maximum or one minimum.*

5. Let us now make a systematic study of all possible histories of the cloud, remembering that, by (14), $\xi = 1$ for $s = 0$.

Case I (*Dense cloud*).

Let us suppose that, for $s = 0$,

$$\frac{\kappa\rho_0}{2\lambda} > 1 \qquad \text{or} \qquad \rho_0 > \frac{2\lambda}{\kappa} . \tag{23}$$

This implies that the line $\xi = 1$ lies to the left of P in figure 1. Since A increases steadily as we increase v^2, holding ξ fixed, it follows that there are the following three possibilities for the future history of a dense cloud.

Ia. If the initial rate of increase of its radius ($\eta_0 v$) exceeds

$$\eta_0(\tfrac{1}{3}\kappa\rho_0)^{1/2}\left[1 + \tfrac{1}{2}\left(\frac{2\lambda}{\kappa\rho_0}\right) - \tfrac{3}{2}\left(\frac{2\lambda}{\kappa\rho_0}\right)^{1/3}\right]^{1/2}, \tag{24}$$

the cloud expands steadily and indefinitely.

Ib. If the initial rate of increase of its radius ($\eta_0 v$) is equal to the critical value (24), the cloud continues to expand, its radius tending asymptotically to the value

$$\eta_0\left(\frac{\kappa\rho_0}{2\lambda}\right)^{1/3}. \tag{25}$$

Ic. If the initial rate of increase of its radius ($\eta_0 v$) is less than the critical value (24), or if its radius is initially stationary or decreasing, the radius of the cloud ultimately tends to zero.

Case II (*Light cloud*).

Let us now deal similarly with the case where

$$\frac{\kappa\rho_0}{2\lambda} < 1 \qquad \text{or} \qquad \rho_0 < \frac{2\lambda}{\kappa} . \tag{26}$$

Now the line $\xi = 1$ lies to the right of P in figure 1, and we have the following three possibilities for the future history of a light cloud.

IIa. If the initial rate of decrease of its radius ($-\eta_0 v$) exceeds the critical value (24), the radius of the cloud tends steadily to zero.

IIb. If the initial rate of decrease of its radius ($-\eta_0 v$) is equal to the critical value (24), the cloud continues to contract, its radius tending asymptotically to the value (25).

IIc. If the initial rate of decrease of its radius ($-\eta_0 v$) is less than the critical value (24), or if its radius is initially stationary or increasing, the cloud expands indefinitely.

Figure 1 is only intended to indicate the general nature of the curves: it is drawn for $\lambda = 3$, $\kappa\rho_0 = 6$, for the sake of simplicity. Thus it actually

corresponds to the case which separates the heavy cloud from the light cloud, but it illustrates the general features of either case.

[1] R. C. Tolman, "Effect of Inhomogeneity on Cosmological Models," *Proc. Nat. Acad. Sci.*, **20**, 169–176 (1934).

[2] Cf. W. Pauli, "Relativitätstheorie," *Encykl. d. Math. Wiss.*, V, **19**, p. 746. I follow Pauli in taking the signature $+ + + -$ for the fundamental form in space-time.

[3] L. P. Eisenhart, *Riemannian Geometry*, p. 113 (1926).

[4] This is the modification to our case, where the line-element is indefinite, of equation (9.23), p. 102, of a paper "On the Geometry of Dynamics," *Phil. Trans. Roy. Soc.*, **A 226** (1926).

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*TO WHAT EXTENT IS DEVELOPMENTAL BLOCK DEPENDENT UPON THE METABOLIC ACTIVITY OF THE EMBRYONIC CELL?**

By Joseph Hall Bodine

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Communicated October 24, 1934

Developmental blocks or diapause in the embryonic life of certain insects are of considerable biological interest since their solution may perhaps give important clues as to the workings of the normal embryonic cell. As experimental materials such forms are of value in that these developmental blocks seem of internal origin and are quite independent of environmental conditions for their occurrence. At the same time it has also been possible to artificially produce developmental blocks resembling, to some degree, those normally found in nature (Slifer;[4] Bodine[2]). Some ideas of the mechanism producing these blocks or diapause in the embryo of the grasshopper, *Melanoplus differentialis*, have recently been advanced (Bodine[1]). It is the purpose of the present paper to show, (1) that the block mechanism in the developing egg of *M. differentialis* is not primarily concerned with the general oxidative metabolism of the cell, and (2) that the oxidation limits of the blocked cell may be modified without interfering in any way with the working of the block mechanism.

Use has been made of the fact that the egg of this form is protected by a semipermeable membrane which normally allows water, but not salts, to pass through freely (Slifer;[4] Bodine[2]). Water removed from the growing embryo results in reversible cessation of cellular activity, lowering of oxygen consumption rates, etc. (Slifer;[4] Bodine[2]). Since cellular activity, such as mitosis, growth, etc., are normally blocked in embryos during diapause (Slifer[4]), it becomes of some interest to determine the extent to which the block mechanism and oxidative metabolism can be influenced by withdrawal and addition of water to the cells. Actively developing and

blocked eggs of the grasshopper, *M. differentialis*, in comparable morphological stages of development (3 weeks at 25°C.), have been subjected to balanced hypertonic solutions and their rates of oxygen intake followed by

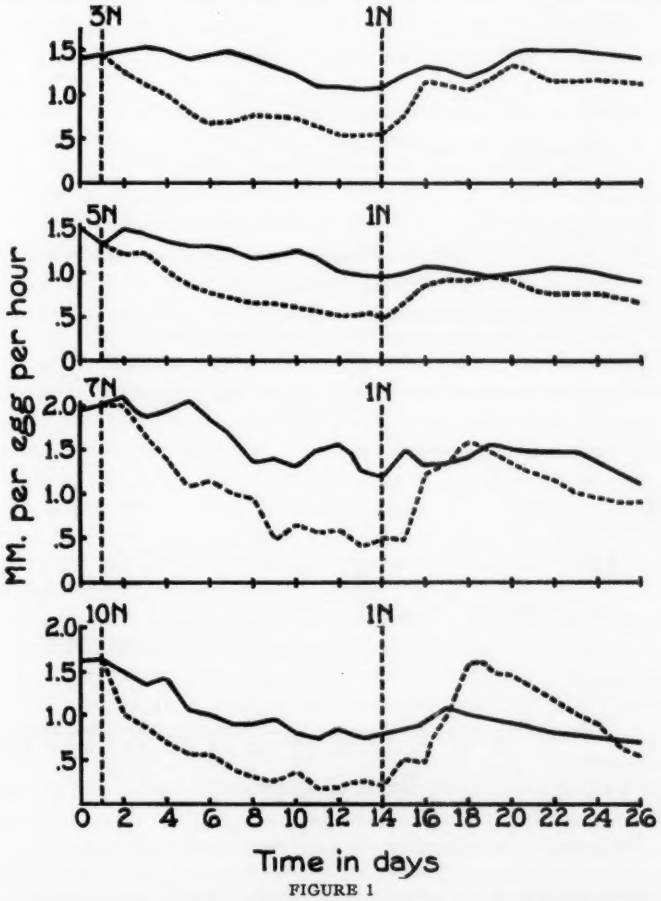

FIGURE 1

Graphs showing effects of different strengths of hypertonic balanced salt solutions on oxygen consumption rates of blocked or diapause eggs. Ordinates, millimeter change of manometer fluid per egg per hour.—Abscissa, time in days indicated. Solid lines, blocked or diapause eggs in normal isotonic solution. Broken lines, diapause eggs in hypertonic balanced salt solutions. Points designated on abscissa indicate time of application of salt solution. For 3 *N* solution, 120 eggs were used; for 5 *N*, 200 eggs; for 7 *N*, 100 eggs; for 10 *N*, 100 eggs. Normal and experimental eggs were run in same manometer. For further explanation see text.

means of the modified Krogh manometer at constant temperatures of 25° and 28°C. The balanced salt solution employed was a modified Ringer as previously described (Slifer;[4] Bodine[2]). Eggs have been used in lots of 5

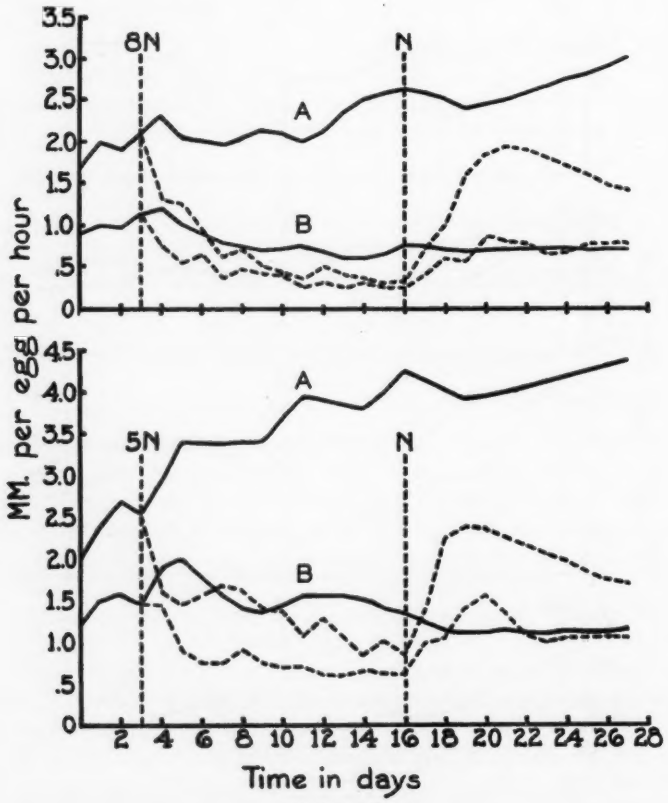

FIGURE 2

Shows effect of hypertonic balanced salt solutions on oxygen consumption rates of developing and blocked embryos. Solid lines (*A*) for developing and (*B*) for blocked eggs in normal isotonic solution. Broken lines for eggs in normal isotonic solution. Broken lines for eggs in hypertonic balanced salt solutions as indicated. For 8 *N* solution, 200 developing eggs and 200 blocked eggs; for 5 *N* solution, 200 developing eggs and 200 blocked eggs. Otherwise same as figure 1.

and 10 and average results expressed graphically since, in general, all experimental results have been qualitatively similar.

It is of some importance to note that in the developing embryos at the 3-

week stage very little if any movement is evident except for a pulsating wave of the lateral body walls (Slifer[3]). In the blocked or diapause embryo no cellular or body activity can be detected. The effects of water loss, as well as changes in oxygen consumption rates, in the two types of embryos cannot be due to any great extent to the cessation of activity as usually found in similar experiments on higher and more fully developed organisms.

Results for blocked or diapause eggs treated with various concentrations of balanced hypertonic solutions are graphically shown in figure 1. From an inspection of this figure it will be seen that as the cells become dehydrated their oxygen consumption rates drop to a more or less constant value depending upon the relative strength of solution employed. The decrease in rate of oxygen consumption is to some extent dependent upon the concentration of the solution employed since a marked drop is noted in going from a 3 N to a 10 N solution as shown in graphs of figure 1. A point of further interest is that the minimum oxygen consumption rate for the egg is relatively quickly reached in the 10 N solution. Reversibility in the diapause egg, as previously pointed out for the developing one (Bodine[2]), is dependent to a great degree upon the length of the period of exposure as well as upon the concentration of the solution applied. It is worthy of note that upon return of treated diapause eggs to normal solutions the oxygen consumption rate returns to the same value as originally given for untreated diapause eggs. This fact is of considerable interest since it affords opportunity to test out and compare the relative effects of dehydration on blocked and developing embryonic cells.

In figure 2 are graphically shown results for developing and blocked eggs exposed to the same strengths of hypertonic solutions. From an inspection of this figure it will be noted that for both normal and blocked eggs the reduction in oxygen consumption rates is approximately to the same level but that when the eggs are again returned to normal solutions recovery of the oxygen consumption rates of the two types of eggs is approximately to their original values. It is, perhaps, a logical conclusion to infer that the block mechanism whatever be its nature, is quite independent of the oxidative mechanism in so far as a decrease in its rate is concerned. One can, perhaps, further infer that the oxygen consumption rate is a result of the block and not a cause. Such a result seems to lend support to recent concepts expressed as to the mechanism of the normally occurring block in the diapause type of egg (Bodine[1]).

In addition it should be pointed out that the results of these experiments seem to show that osmotic effects can definitely be shown to occur in the oxygen consumption rates of cells in which body movement, mitosis, etc., are almost completely absent.

Summary.—(1) Studies have been made on the action of hypertonic

balanced salt solutions on the oxygen consumption rates of blocked or diapause eggs of the grasshopper, *Melanoplus differentialis*.

(2) The block or diapause mechanism in the developing egg of this form is not primarily dependent upon the general oxidative metabolism of the cell.

(3) Changes in the rates of oxygen consumption of the blocked or diapause cell are possible without interfering in any way with the working of the block mechanism.

* Acknowledgment is made to the Rockefeller Foundation for grant for work on Cellular Physiology.

[1] Bodine, J. H., *Physiol. Zoöl.*, **5**, 549–554 (1932).

[2] Bodine, J. H., *Ibid.*, **6**, 150–158 (1933).

[3] Slifer, E. H., *Biol. Zentralbl.*, **52**, 223–229 (1932).

[4] Slifer, E. H., *J. Exp. Zoöl.*, **67**, 137–157 (1934).

A THEORETICAL BASIS FOR INTENSITY DISCRIMINATION IN VISION

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Communicated October 19, 1934

The problem of intensity recognition and discrimination, though recognized and approximately formulated by Bouguer[1] in 1760, received its first quantitative statement by Fechner in 1860 on the basis of the work of Weber in 1834. If the prevailing intensity of a visual field is I, and that of a just perceptibly brighter portion of it is $I + \Delta I$, then Fechner assumed that the ratio $\Delta I/I$ remains constant regardless of the value of I. Aubert[2] was the first to show by measurements that this assumption is contrary to fact, and his results have been corroborated by a variety of workers during the last seventy-five years. Instead of $\Delta I/I$ being constant, it is found to vary regularly from as much as $^1/_3$ at low intensities to as little as $^1/_{167}$ at high intensities.

The classic research on the way in which $\Delta I/I$ varies with I has been the work of Koenig and Brodhun.[3] They found, as had Aubert, that as I increased, $\Delta I/I$ decreased. In addition, they found that with further increase in intensity $\Delta I/I$ again rose. This rise at high intensities has been generally accepted, and has formed a critical part of the theoretical explanations which first Pütter[4] and later I[1,5] gave for the phenomenon and which have recently been reiterated by Houstoun.[6]

In the last two years the picture has changed fundamentally with respect to this rise in the fraction $\Delta I/I$ at high intensities. In the first place, the

intensity discrimination of the bee as measured by Wolf,[7] and of *Drosophila* as measured by Wald and myself,[8] shows no rise in $\Delta I/I$ at high intensities. Instead, as the intensity increases, $\Delta I/I$ decreases steadily to a minimum at which it remains constant. Wald and I paid particular attention to the possibility of this rise, but though we tested intensities even 10,000 times higher than the one at which the minimum becomes established, we found no trace of an upturn in $\Delta I/I$.

All this might merely mean that the basis for intensity discrimination in the bee and the fruit-fly is different from that in the human eye were it not for the second fact that the validity of the upturn for the human eye itself has also been questioned. Though he records no measurements, Guild[9] has published an experiment which shows that the rise in $\Delta I/I$ at high intensities is "entirely factitious and depends on the degree of adaptation to each field-intensity which prevails when the observation is made." Independently, Steinhardt, working in our Laboratory, has shown that the appearance of a rise in $\Delta I/I$ depends on the illumination surrounding the test-field and on the adaptation of the eye. The better the adaptation, the less evident is the upturn, and when adaptation is complete there is little or no upturn at high intensities especially if the test area possesses a large surrounding field of the same brightness as the test-field. Koenig and Brodhun had no surround to their test-field.

Since the upturn in $\Delta I/I$ is so integral a part of the previous theories of intensity discrimination, its non-existence in the insect eye and in the human eye renders a new formulation necessary for the data. We may begin with the insect eye because the experimental conditions are clearly defined for theoretical purposes.

The measurements with insects were made by confronting them with a visual field composed of stripes which, when moved, elicit a movement of the animal opposite in direction to the stripe displacement. The stripes can radiate any fraction of the light coming through the clear spaces, and the measurements consist in finding the least difference ΔI between the light intensity I coming from the stripes and the intensity $I + \Delta I$ from the clear spaces, which will just barely elicit a response from the insect when the pattern is moved.

In the measurements the response of the animal follows immediately upon the movement of the pattern. Thus, though a given group of ommatidia have been adapted to the intensity I when the stripes are at rest, the immediate response of the insect when the stripes are moved precludes the adaptation of these ommatidia to the higher intensity $I + \Delta I$. The change in the ommatidia which initiates the events resulting in a response must therefore take place immediately when the outside light is changed from I to $I + \Delta I$, and it is to this initial event that one must look for an explanation of the data of intensity discrimination.

The process of photoreception, considered in the most general terms, requires the presence of (*a*) an inactive photosensitive substance which absorbs light and is changed by it into an active substance responsible for initiating the train of events which ends in a nerve impulse, and (*b*) some means of maintaining a supply of the sensitive material, since otherwise it would be used up and the process would come to an end. Obviously the photoreceptor system is more complicated than this; but, fortunately, we need not consider anything beyond this very first step in order to describe the data.

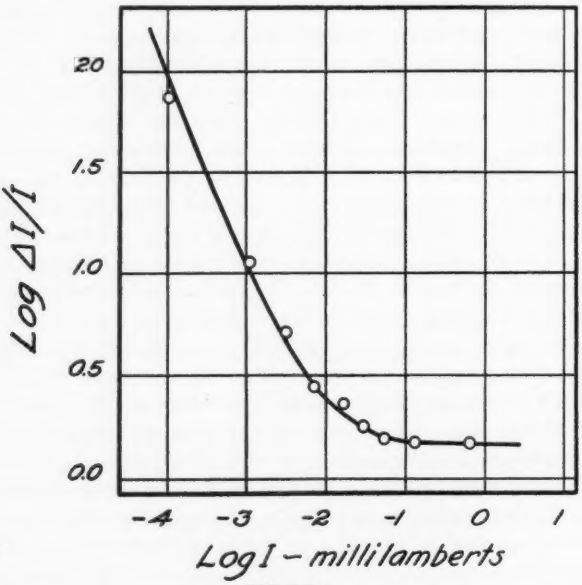

FIGURE 1

Measurements of intensity discrimination of *Drosophila*. The theoretical curve is from equation (6).

In such a photochemical system, the sensitive material S is changed by light into the photoproducts $P, A, B, \ldots$ some of which may under proper conditions reunite to form the sensitive material from which they were derived. Let the total initial concentration of S be a, and the concentration of $P, A, \ldots$ at the moment t be x. The velocity of the process as a whole under the influence of light of intensity I will be

$$dx/dt = k_1 I(a - x)^m - k_2 x^n \qquad (1)$$

where m and n represent the order of the photochemical and the dark

reaction respectively, and k_1 and k_2 are their velocity constants, k_1 including the absorption coefficient. If the light continues to shine, a stationary state is reached in which the concentrations of sensitive material and decomposition products remain constant, so that $dx/dt = 0$, and

$$k_1 I(a - x)^m = k_2 x^n. \qquad (2)$$

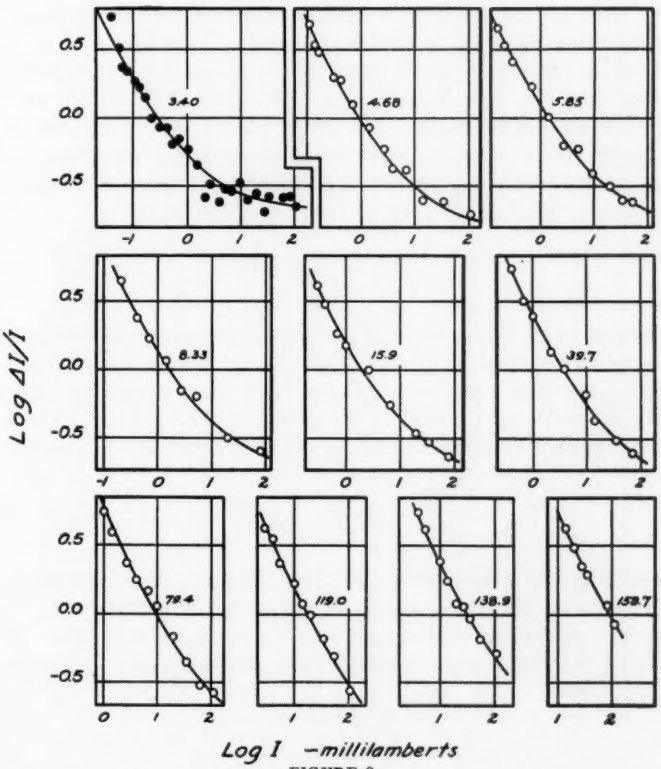

FIGURE 2

Wolf's measurements of the intensity discrimination of the honey bee. The black circles are the data from the first paper; the plain circles from the second paper. The numbers attached to the curves are the visual acuities multiplied by 1000 and are inversely proportional to the size of the stripes used for the measurements. The same theoretical curve is drawn through all the data and is from equation (7).

If the photosensory system at the stationary state induced by I is now exposed to light of intensity $I + \Delta I$, the velocity at the first instant of exposure will be

$$dx/dt = k_1(I + \Delta I)(a - x)^m - k_2x^n. \tag{3}$$

Since they are equal, we may add $k_1I(a - x)^m$ to and subtract k_2x^n from the right side of equation (3), with the result that it becomes

$$dx/dt = k_1\Delta I(a - x)^m. \tag{4}$$

For *Drosophila* and the bee, the sensory recognition of the addition of ΔI occurs immediately after the increase is made. Therefore, some property of the initial velocity (4) is the basis for the recognition of the difference between I and $I + \Delta I$. Assume that for the intensity $I + \Delta I$ to be distinguished from the intensity I, the initial velocity on the addition of ΔI must be the same no matter what the intensity I may have been. This means $k_1\Delta I(a - x)^m = c$, where c is a constant. Taking the value of ΔI from this equation, and the value of I from equation (2), we get

$$\frac{\Delta I}{I} = \frac{c}{k_2} \cdot \frac{1}{x^n} \tag{5}$$

as a description of $\Delta I/I$ in terms of the general ideas just considered.

In order to test these ideas with data, it is necessary to eliminate x from (5) and to substitute values of I. To do this we must try specific values for m and n in equations (2) and (5). When $m = n = 1$, that is when both the light and the dark reactions are monomolecular, equation (5) becomes

$$\frac{\Delta I}{I} = \frac{c}{ak_2}\left(1 + \frac{1}{KI}\right) \tag{6}$$

where $K = k_1/k_2$. Plotted as log $\Delta I/I$ against log I, the form of the function is independent of the constants K and c/ak_2. K controls the position of the curve on the axis of abscissas; c/ak_2 on the axis of ordinates. When $m = n = 2$, that is when both reactions are bimolecular, equation (5) becomes

$$\frac{\Delta I}{I} = \frac{c}{a^2k_2}\left(1 + \frac{1}{(KI)^{1/2}}\right)^2 \tag{7}$$

which may also be plotted in logarithmic form so that its precise shape is independent of the constants K and c/a^2k_2. The unsymmetrical case of $m = 1$ and $n = 2$ gives

$$\frac{\Delta I}{I} = \frac{c}{k_2} \cdot \frac{4}{[(4aKI + K^2I^2)^{1/2} - KI]^2} \tag{8}$$

which may be plotted logarithmically in the same way as (6) and (7). In all cases the total concentration of sensitive material is put at 100 per cent; thus $a = 1$.

The theoretical behavior of $\Delta I/I$ in all three equations resembles the actual behavior of $\Delta I/I$ for insects. The data for *Drosophila* are shown in figure 1. Through them is drawn the curve for $\Delta I/I$ in accordance with equation (6). The fit with this equation is slightly, though not decisively,

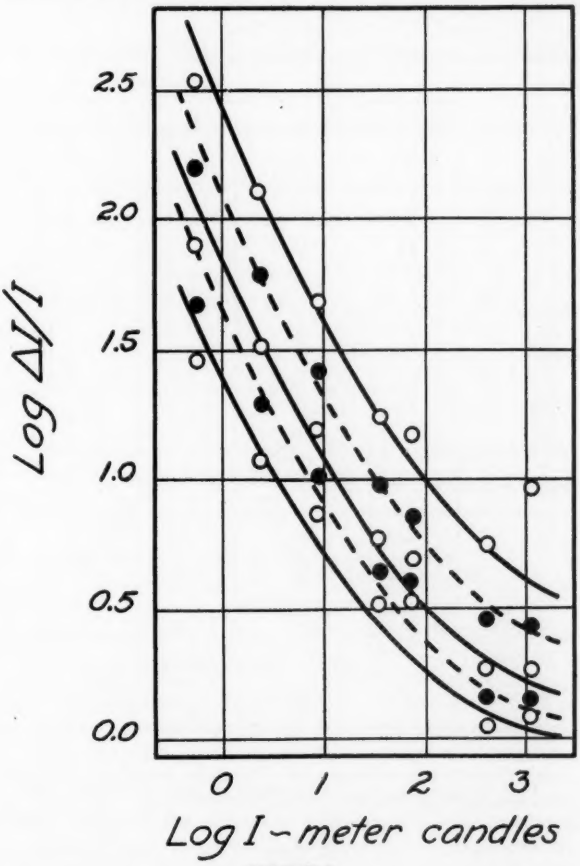

FIGURE 3

Intensity discrimination of the clam, *Mya*. The clear circle so obviously off is a value extrapolated from the measurements and therefore unreliable. The curves are all from equation (7).

better than with equation (8); equation (7), however, is definitely ruled out.

Wolf's first measurements of the bee's intensity discrimination are shown in figure 2 by black circles. They are too scattered to provide a critical

choice among the three equations. Wolf's later measurements, shown by clear circles, are smoother, more numerous and more critical. Of the three theoretical curves, only the one from equation (7) can be drawn through them. The goodness of fit is obvious. Because of the adequacy of equation (7) for the later data, the same curve is drawn through the earlier.

Drosophila and the bee have organized eyes. The clam, *Mya*, on the

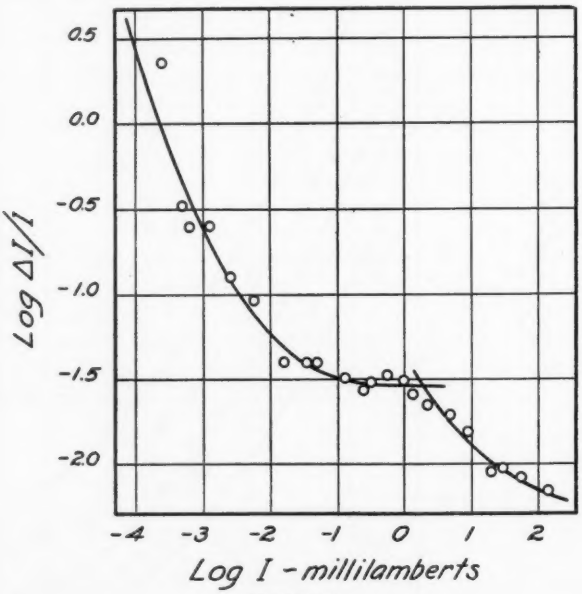

FIGURE 4

Aubert's measurements of the intensity discrimination of his own eye. The original intensities have been divided by 10,000 to convert them into millilamberts. The data strikingly break into two sections representing rod and cone functions. The curve for low intensities is from equation (8), and the one for high intensities is from (7).

other hand, has a diffuse sensitivity to light all over its siphon. Does the present formulation apply to it? The measurements with *Mya*, made ten years ago,[10] record the necessary increase in illumination to which the animal responds immediately with a specific reaction time. Figure 3 shows the data for responses at five different reaction times, from which it is clear that the relationship of $\Delta I/I$ to I is the same for all reaction times. The curves are from equation (7), which is the only one of the three to fit.

Equation (7) assumes $n = m = 2$. That $n = 2$ for *Mya* is not surprising since the dark reaction has long been known to be bimolecular;[11] that the purely photochemical reaction is also of the second order emerges for the first time from this theoretical treatment.

The success of the proposed theory of intensity discrimination with *Drosophila*, the bee, and *Mya*, suggests its consideration for the human eye. One difficulty becomes at once apparent. Measurements of intensity discrimination with the human eye are made with a bipartite test-field of

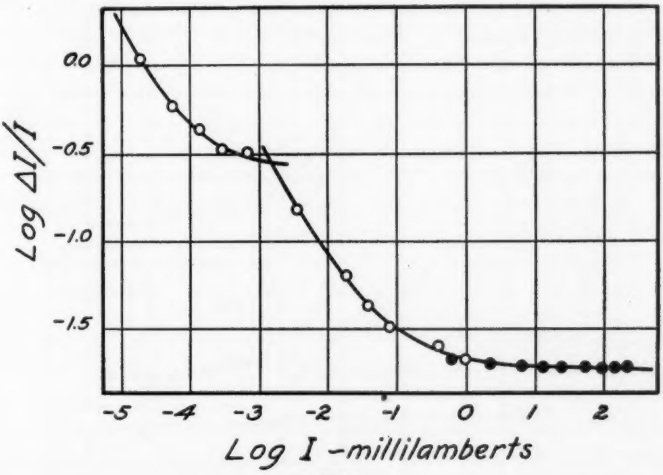

FIGURE 5

The measurements of Blanchard are the plain circles; those of Lowry are the black circles. Lowry's points have been raised 0.15 log unit along the ordinates to make them continuous with Blanchard's. The curves are the same theoretical ones as for Aubert's data in figure 4. Note here too the natural breaking of the data into two sections indicative of rod and cone functions.

which one side has an intensity I and the other an intensity $I + \Delta I$. Such a procedure seems to involve the simultaneous and complete adaptation of the corresponding retinal areas to their respective intensities. In fact it was on the supposition of such complete adaptation that ten years ago[1] I formulated an explanation of visual intensity discrimination which includes an upturn of $\Delta I/I$ at high intensities. The present formulation, however, relies on the initial effect of the added intensity ΔI and not on the final effect after complete adaptation, as would apparently be required by the experimental procedure.

This antithesis may, however, be more apparent than real. In making the measurements, the eye looks at and becomes adapted to the intensity I

which prevails on both sides of the field. Then the intensity on one side is raised slightly, and a judgment is made as to whether there has been a perceptible increase in brightness. The procedure is continued until the minimum increase is found which is clearly recognizable not only at once but on continued examination. It is to be noted, however, that any continued study of the field involves the usual and persistent eye movements

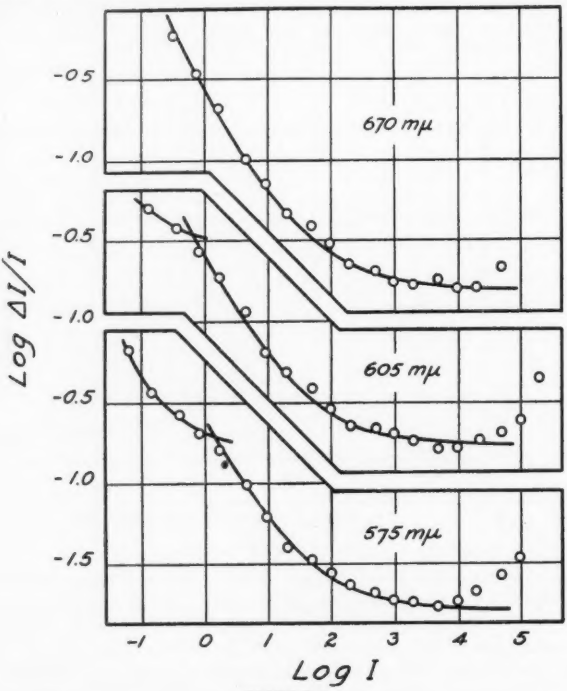

FIGURE 6

The data of Koenig and Brodhun for Koenig's eye for red, orange and yellow spectral light. The red data are continuous and show only cone function, whereas the orange and yellow show increasing amounts of rod function. The curves are theoretical and are from equation (8) for the rod section and from (7) for the cones.

which expose fresh portions of the retina to the higher intensity, while the general state of adaptation of the retinal field still corresponds to the lower intensity. If this is correct, then recognition of ΔI may involve initial rather than final effects even with the human eye—a point of view which would be strengthened if the data could be described in terms of the equations just derived.

The oldest data of intensity discrimination are for Aubert's eye, and his single, individual measurements are shown in figure 4. The data range themselves into two parts which are a clear demonstration of the then non-existent duplicity theory, in terms of which the low intensity part represents the function of the rods, and the high intensity part that of the cones. The two sections of the data are fitted by the equations of the present theory. The low intensity data are not good enough for a choice between equations (6) and (8), but they are adequate tentatively to rule out equation (7). Equation (8) has been drawn through them, because the dark reaction of the rods is probably bimolecular.[12] The high intensity portion is too limited for a critical choice among the equations; the curve drawn is from equation (7), but this is on the basis of the other measurements to be considered.

Blanchard's data,[13] shown as plain circles in figure 5, were made about fifty years after Aubert's. They do not cover quite so large a range as Aubert's, but the points are averages. These data also break by themselves into two portions. The curve through the rod portion is again from equation (8), though all three curves are indistinguishable over such a small range. Blanchard's cone data may be supplemented by Lowry's[14] measurements made in the same laboratory thirteen years later. They cover the higher intensities and are the black circles in figure 5. The curve from equation (7) is the only one which passes through the points.

These measurements are with white light, and show the separate presence and function of the rods and cones. Figure 6 contains the data of Koenig and Brodhun with the red, orange and yellow light. Since the extreme red of the spectrum even at low intensities is more effective for the cones than for the rods, it is not surprising to find that the points for 670 mμ lie on one continuous curve and show no trace of the break so strikingly present with white light. The data again are fitted only by equation (7). The data for 605 mμ and for 575 mμ show the usual discontinuity and the separate presence of rod and cone function. As would be expected from the relative effectiveness of 605 and 575 mμ at low intensities for rods and cones,[15] the rod portion for 575 mμ is larger than for 605 mμ. The cone portions of 605 and 575 mμ are again fitted only by equation (7). In all cases, the few points at high intensities must be disregarded for the reasons already given.

Koenig and Brodhun's data with white light resemble neither Aubert's nor Blanchard's, nor indeed their own data with orange and yellow light, particularly the latter which usually produces effects very similar to white light. They show no clear division into two parts, and though the points fall around two theoretical curves such as are shown in figure 5, they are not really fitted by them. Some condition in their procedure with white light

must be responsible for the difference, but unfortunately Koenig and Brodhun published only the scantiest technical details, and leave one at a loss to know what this condition is.

Steinhardt's measurements, made in our Laboratory and soon to be published, confirm this conclusion about Koenig and Brodhun's white light data. Steinhardt measured the relation of $\Delta I/I$ to I for different test-field sizes. For white light and test areas larger than 2° his measurements, without exception, fall on a double curve similar to the data of Aubert and of Blanchard, while for smaller, foveally fixated areas, they always form single curves like those of Koenig and Brodhun with light of 670 mμ. With Steinhardt's permission I give in figure 7 two examples of his many series. The upper data are for white light and a test area 56′ in diameter, having a large surround in order to maintain the eye as a whole at the

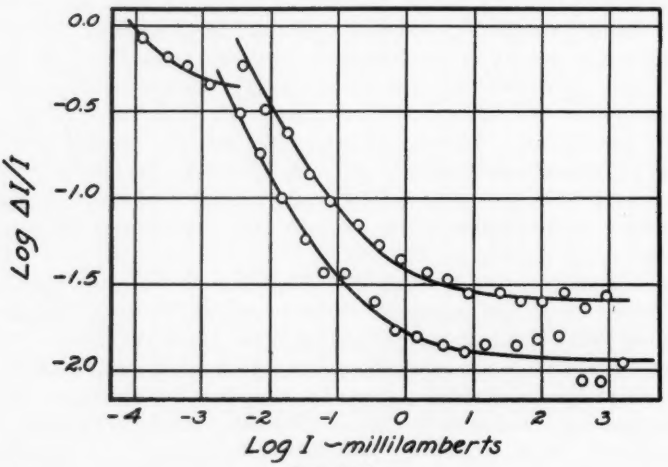

FIGURE 7

Steinhardt's measurements with white light. The upper data are with a field 56′ in diameter; the lower with a field 3° 44′ in diameter. The upper data show only cone function and are described as usual by the curve from equation (7). The lower data show both rod and cone function; the curve through the former is from equation (8), while through the latter it is from (7).

intensity of the measurements. This size of test-field falls entirely within the rod-free area of the fovea, and as a result the measurements show no inflection point. The only curve which passes through the points is again from the cone equation (7). The lower data in figure 7 were made with a field size nearly the same as Koenig and Brodhun's. The measurements happen to be some of Steinhardt's earliest, but like all others, they clearly resemble the white light data of Aubert and of Blanchard, and not of

Koenig and Brodhun. The curves drawn through them are from equation (8) for the short piece at low intensities, and from equation (7) for the rest of the data. Therefore, all the reliable data on the human eye are fitted by the equations derived from the theory.

A paper dealing in detail with the theory and the measurements is to appear in the *Journal General Physiology*.

Summary.—A theory of visual intensity discrimination is proposed which describes quantitatively all the established, available data from the photosensory systems possessed by such diverse animals as the clam, *Mya;* the insects, *Drosophila* and the bee; and the human eye. The theory depends on the photochemical events which take place at the moment when a photosensory system already adapted to the intensity I is exposed to the just perceptibly higher intensity $I + \Delta I$. Unlike previous formulations, this theory predicts that the fraction $\Delta I/I$, after rapidly decreasing as I increases, does not increase again at high intensities, but reaches a constant value which is maintained even at the highest intensities.

[1] For a summary of the history of this subject see Hecht, S., *J. Gen. Physiol.*, **7**, 235 (1924).

[2] Aubert, H., *Physiol. d. Netzhaut*, 394, Breslau (1865).

[3] Koenig, A., and Brodhun, E., *Sitzber. d. Akad. d. Wiss.*, 917, Berlin (1888); 641 (1889).

[4] Pütter, A., *Arch. ges. Physiol.*, **171**, 201 (1918).

[5] Hecht, S., *J. Gen. Physiol.*, **11**, 255 (1928).

[6] Houstoun, R. A., *Report of a Joint Discussion on Vision*, Physical Society, 167, London (1932).

[7] Wolf, E., *J. Gen. Physiol.*, **16**, 407, 773 (1933).

[8] Hecht, S., and Wald, G., *Ibid.*, **17**, 517 (1934).

[9] Guild, J., *Report of a Joint Discussion on Vision*, Physical Society, 180, London (1932).

[10] Hecht, S., *J. Gen. Physiol.*, **6**, 355 (1924).

[11] Hecht, S., *Ibid.*, **1**, 545 (1918).

[12] Hecht, S., *Ibid.*, **2**, 499 (1920).

[13] Blanchard, J., *Phys. Rev.*, **11**, 81 (1918).

[14] Lowry, E. M., *J. Opt. Soc. Amer.*, **21**, 132 (1931).

[15] Hecht, S., and Verrijp, C. D., *Proc. Nat. Acad. Sci.*, **19**, 522 (1933).

THE DEVELOPMENT OF RAT EMBRYOS IN TISSUE CULTURE

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Explantation methods have shown inconstant results when applied to the problems of mammalian development. The results obtained with rabbit material have so far been superior to those secured in other forms. The rabbit egg[1,2,3] can be carried through the first stages of development outside the body with normal development. The eggs so treated may be tested[4] by reimplantation in the uterus with the development to birth of normal rabbit young. Later stages of rabbit embryos (early somite) have been less adaptable to explantation methods.[5,6,7] The limit of development has been approximately 48 hours with degeneration phenomena appearing in the central regions.

In the rat, development of the egg outside the body has been limited to one or two cell divisions;[8] most of the work therefore consists of observations upon uncultured material.[9,10] At first glance it might seem that the rat egg lacks developmental potency under changed conditions but recent studies[11] have shown that the egg will develop into an embryo in the abdomen, that it will differentiate into embryonic tissues under the kidney capsule,[12,13] and that each of the first two blastomeres has the capacity for forming a complete embryo which, however, does not develop[14] beyond the egg cylinder stage.

While the tissue culture methods so far used seem inadequate in the study of segmenting stages, they have proved much more successful for the later embryonic series. Our first experiments comprised a long series of embryos at stages varying from the early egg cylinder (ca. 6–7 days) to embryos of 20 somites (ca. 11 days). These were explanted to various media (chick plasma, starch agar, nutrient agar, placental extractions, agar + varying salt concentration, chick plasma + rat embryonic extract) with no greater success than recorded for the rabbit.

Experiments made upon the rigidity of the clot showed that a jell composed of chick plasma and rat embryo extract afforded a better substrate from the mechanical standpoint than any of the other combinations. In all possible combinations, however, the embryo was being studied under survival conditions and necrotic areas were prominent after 6 hours in some cases, 12 hours in most and in all at 24 hours. Necrosis is not limited to one specific region of the embryo nor to one tissue. When it occurs the differentiation of adjacent parts is not seriously affected initially. Necrotic areas are readily observed in the living culture and show as whitened opacities in the embryo.

The present report is based on about one hundred successful experiments in which development was observed from presomite or early 5–7 somite stages after 24, 48 and 72 hours of explantation. The medium used was heparinized rat plasma plus rat embryonic extract prepared in general from embryos of ca. 14–15 days. In preparing the extract, the embryo, freed from its membranes and placenta, is washed after which it is minced and centrifuged; the fluid is discarded, fresh sterile Ringer added and again centrifuged for usable extract. The rat plasma is obtained from adults of either sex. The two ingredients are mixed in approximately equal amounts in embryological watch glasses just before the embryo is transplanted.

The embryos for transplantation are removed from the uterus with decidua intact. The decidua is then carefully stripped from the embryo and discarded; the embryo is washed in two changes of sterile Ringer's solution, transferred to the plasma with as little excess of Ringer's solution as possible and an additional covering of embryonic extract added.

The embryonic material at the time of transplantation consists of a cylindrically shaped rod of material. The embryo occupies roughly the upper third; the lower third is the träger or ectoplacental cone, while the middle third is the so-called extra-embryonic space which is later condensed into the basal portion at the time of placenta formation, due to the growth and expansion of the embryo and the amniotic sac. The embryonic cylinders were carefully dissected from the decidua without damaging the covering of the cylinder (Reichert's membrane). The embryos were fixed at 24, 48 and 72 hour intervals, after frequent observation during their cultivation, 37.5°C. The embryos were sectioned in the clot and compared with equivalent stages of normal development.

The results in general are striking. Growth occurs during the first 24 hours but gradually slows until 36 hours when it practically stops. The majority of our cases showed steady differentiation. At 48 hours or earlier, differentiation in the embryo has reached a maximum, at which it may be maintained for another 24 hours with the present method.

During this period the embryos in our best cases have differentiated from 2 to 16 somites. The allantoic bud has grown from a small lump of tissue at the angle between the amnion and the posterior part of the embryo to join with the superior surface of the ectoplacental cone. The heart, unformed at the time of implantation, has differentiated a two chambered structure and has initiated its beat, the blood islands have developed in the yolk sac epithelium, and circulation has commenced, both in the yolk sac and in the embryo. The nervous system has differentiated considerably; eyes and ears have differentiated and the embryo as a whole has gone through a primary torsion, separating it from the

embryonic membranes in the region of the intestinal portal and contributing to its apparent reversal of posture.

The total growth attained in the 48 hour period is less than half that attained by the normal embryo during the same period. The maximum differentiation is nearly three-quarters of that undergone by the normal. The factors limiting growth are affected earlier than those limiting differentiation.

Apparently respiratory interchange is the most important functional necessity at this stage. The efficiency of this mechanism is not only lowered by the total absence of maternal circulation but even further prevented by the growth of a new enveloping membrane in the nature of a decidua from the marginal cells of the ectoplacental cone. The accumulation of break-down products due to metabolic activity is another checking factor. A few preliminary experiments have shown that these can be removed by washing the entire culture in sterile Ringer's solution and adding fresh embryonic extract. By using this method embryos have been kept alive for 96 hours although growth and differentiation occur only at a low rate during the last 24 hours.

These experiments show definitely (1) that the processes of differentiation and growth can be artificially separated by explantation of an embryo to tissue culture; (2) that the differentiation of embryonic structures in normal relationship to each other can successfully be completed outside the body; and (3) that the method offers an opportunity for the study of hitherto obscure processes such as the turning of the embryo, the initiation of circulation, the physiological reactions of the embryo during critical stages of mammalian development.

[1] Brachet, A., *Arch. d. Biol.*, **28**, 447–503 (1913).
[2] Lewis, W. H., and Gregory, P. W., *Anat. Rec.*, **42**, 27 (1929).
[3] Pincus, G., *Proc. Roy. Soc.*, **B107**, 132–167 (1930).
[4] Pincus, G., and Enzmann, E. V., *Proc. Nat. Acad. Sci.*, **20**, 121–122 (1934).
[5] Waterman, A. J., *Anat. Rec.*, **54**, 72 (1932).
[6] Waterman, A. J., *Proc. Nat. Acad. Sci.*, **20**, 145–147 (1934).
[7] Waddington, C. H., and Waterman, A. J., *Jour. Anat.*, **67**, 356–370 (1933).
[8] Defrise, Aldo, *Anat. Rec.*, **57**, 239–250 (1933).
[9] Long, J. A., *Univ. Calif. Publ. Zoöl.*, **9**, 105–136 (1912).
[10] Gilchrist, F., and Pincus, G., *Anat. Rec.*, **54**, 275–288 (1932).
[11] Nicholas, J. S., *Ibid.*, **58**, 387–413 (1934).
[12] Nicholas, J. S., and Hall, B. V., *Ibid.*, **58**, 83 (1934).
[13] Buyse, A., *Ibid.*, **55**, 10 (1933).
[14] Nicholas, J. S., *Ibid.*, **55**, 31 (1933).

EYE-STALK HORMONE AND THE MOVEMENT OF DISTAL RETINAL PIGMENT IN PALAEMONETES

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As is known from investigations by Parker ('97) and by Welsh ('30), the distal pigment cells in the eyes of Palaemonetes undergo positional changes in adaptation to light and dark. Parker's work ('97) further gave rise to the debatable point as to whether the retinal-pigment migration in the arthropods is a direct response of the cells to light or is more in the nature of a reflex involving the central nervous system.

This question recalls the problem of the interrelationship of the eyes; if the retinal pigment migration is independent of the central nervous system, then the photomechanical condition of the retina in one light-stimulated eye of an animal should have no effect upon the condition in the other covered eye of the same animal; if, on the other hand, the retinal-pigment migration involves the nervous system, then the two eyes may be "sympathetic," that is, the photomechanical condition of one eye may influence the other eye.

The evidence presented on this subject has been inconclusive and uncertain. Parker's experiments ('97) led him to conclude that the photomechanical condition of the retina of one eye had no effect on that of the other eye, and that in excised eyes complete or nearly complete photomechanical changes may occur, thus indicating that the retinal pigment migration is independent of the brain and optic ganglia. His conclusions that these pigment changes are independent of the central nervous system are open to criticism since they are based on the reactions of only 16 animals in four experimental conditions. Von Frisch ('08), in turn, could get no decisive results in his experiments on Palaemon in which one eye was illuminated and the other covered. Castle ('27), who studied the proximal pigment in Palaemonetes for a possible interrelationship of the two eyes, came to the conclusion "that in Palaemonetes an illuminated eye is without 'sympathetic' effect by way of the central nervous system on the photomechanical changes of the covered eye."

Perkins in 1928 found the body chromatophores of Palaemonetes to be under control of the eye-stalk hormone, and it has since been suggested by Parker ('32) and by Bennitt ('32) that there might be a hormone factor in the migration of the retinal pigment. Of the three different kinds of pigment cells in the eye-stalk of Palaemonetes, distal, proximal and accessory, this paper is concerned only with the distal cells. For the sug-

gestion of this problem and for interest and aid in the preparation of this manuscript I am indebted to Professor G. H. Parker.

Three series of tests were employed in the experimental procedure. In the first series the stalk extract was prepared from the eyes of shrimps that had been in the dark room overnight, a concentration equivalent to 100 stalks in 1 cc. of sea water being used. Readings of the position of the pigment in the eyes of light-adapted animals were made according to the method of Welsh, and then 0.03 cc. of the eye-stalk solution were injected into the abdominal musculature. Observations were made through the ocular micrometer at 3 or 5 minute intervals over a period of 30 to 60 minutes. In 8 shrimps thus tested there was no obvious change in the position of the pigment. Some readings showed a slight variation, ranging from one-half to one unit on the micrometer scale (approximately 8 to 16 micra) but since the normal migration, according to Welsh, ranges from 140 to 220 micra, these slight differences were regarded as of no significance.

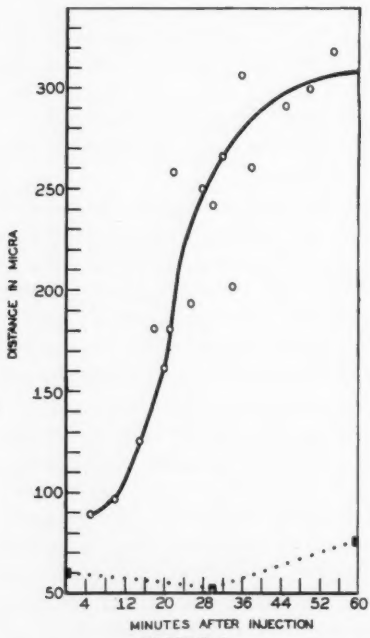

FIGURE 1

Graph showing in solid line the amount of inward migration of the distal pigment cells in the eyes of dark-adapted Palaemonetes as a result of injections, in the dark, of stalk-extract prepared from the eyes of light-adapted shrimps. The dotted line shows the position of the distal pigment in the eyes of three uninjected control animals.

In the second series of tests, light-adapted animals were used as before, but the eye-stalk solution was prepared from shrimps kept over a black background. Concentrations equivalent to 100 and to 160 stalks for each cubic centimeter of sea water were injected in small quantities (0.02–0.03 cc.) into eight animals. Readings were taken at 5 to 10 minute intervals for 30 to 60 minutes. No regular migration of the pigment was observed other than the slight variation noted before.

In the third series of tests the procedure was reversed, the eye-stalk hormone being prepared from light-adapted shrimps and being injected, in the dark, into animals that had been in the dark room overnight. Using

concentrations of 50 to 60 eye-stalks to 1 cc. of sea water, amounts ranging from 0.02 to 0.05 cc. were injected into several shrimps and individuals were killed at two minute intervals in hot water to fix the retinal pigment-cells. After this preliminary fixation the eye-stalks were excised, dehydrated and cleared in cedar-oil. Readings were then taken of the position of the distal pigment-cells by means of the micrometer eye-piece. A gradual retreat of the pigment-cells was observed to have taken place, the maximum effect being noted 35 to 45 minutes after the injection. This was repeated with ten groups of animals, each group consisting of 10 to 20 individuals, and in each case the response was essentially similar.

The measurements obtained from the animals in one such group give the curve shown in figure 1, where the time in which the hormone was allowed to act after injection is plotted against the distance of the distal ends of the pigment-cells from the outer surface of the eye.

As a control experiment three groups of dark-adapted shrimp were injected: in the first group were eleven animals injected with 0.01 to 0.05 cc. of sea water and fixed at two minute intervals; in the second group were twelve animals injected with 0.02 to 0.04 cc. of an extract of the abdominal musculature of three light-adapted shrimps in 1 cc. of sea water and fixed at five minute intervals; and in the third group were fifteen animals injected with 0.02 to 0.04 cc. of sea water and fixed at five minute intervals. In none of these control animals was there any significant migration of the distal pigment cells. There was a slight change of 1 to 2 units (16 to 32 micra) in some animals, but this was without doubt due to individually varying factors such as the size of the eye and the positions of the pigment-cells during dark-adaptation.

It is evident from these experiments that the eye-stalks of Palaemonetes, when stimulated by light, secrete a substance which causes the inward migration of the distal pigment-cells, but that no substance, opposite in action, is secreted in the dark, so that outward movement of the distal pigment-cells is not dependent upon a hormone but must be inherent in the pigment-cell itself.

Bennitt, R., "Physiological Interrelationship in the Eyes of Decapod Crustacea," *Physiolog. Zoöl.*, **5**, 49–64 (1932).

Castle, E. S., "The Interrelation of the Eyes of Palaemonetes as Concerns Retinal Pigment Migration," *Proc. Nat. Acad. Sci.*, **13**, 637–639 (1927).

Von Frisch, K., "Studien über die Pigmentverschiebung im Facettenauge," *Biol. Zbl.*, **28**, 662–671, 698–704 (1908).

Parker, G. H., "Photomechanical Changes in the Retinal Pigment Cells of Palaemonetes, and Their Relation to the Central Nervous System," *Bull. Mus. Comp. Zoöl.*, **30**, 273–300 (1897).

"The Movements of the Retinal Pigment," *Ergeb. Biol.*, **9**, 239–291 (1932).

Perkins, E. B., "Color Changes in Crustaceans, Especially in Palaemonetes," *Jour. Exp. Zoöl.*, **50**, 71–105 (1928).

Welsh, J. H., "The Mechanics of Migration of the Distal Pigment Cells in the Eyes of Palaemonetes," *Jour. Exp. Zoöl.*, **56**, 459–494 (1930).

THE DISSECTION OF CLOSED SURFACES AND THE PHRAGMEN-BROUWER-ALEXANDROFF THEOREM

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We have recently proved the following theorems[1] in which F denotes the common boundary of two domains in an n-dimensional Euclidean space R_n.

I. If Z is a cut on F then there lie on F arbitrarily small $(n-1)$-dimensional Cantor manifolds which meet Z.

II. On any cut of F lies a point which is the limit of a decreasing sequence of $(n-1)$-dimensional Cantor manifolds lying on F.

Such a point we call[2] a concentric $(n-1)$-dimensional manifold point (Mannigfaltigkeitspunkt) of F. The essential point in these theorems is not the mere existence on F of arbitrarily small Cantor manifolds but a certain connectivity in their distribution on F. For example it follows from II that the set of manifold points of F is by itself a connected set and therefore at least 1-dimensional. The question of the higher connectivity of this set we shall discuss in a later paper in which we shall establish a certain duality in the distribution on F of Cantor manifolds of different dimensions. Our object in the present paper is to prove two theorems (called I* and II* in §3) which include Theorems I and II but give a deeper insight into the structure of the surface F.

Our results are closely related to the generalized Phragmen-Brouwer Theorem which states[3] that if neither of the closed point-sets F' and F'' divides the space R_n then the sum $F'+F''$ divides the space if and only if the intersection $F'F''$ contains an $(n-2)$-dimensional Vietoris cycle homologous to 0 on F' and F'' but not on $F'F''$. The following theorem follows at once from our results:

Let F' and F'' be two closed point-sets in R_n and suppose that there are two points P', P'' which are separated by the sum $F'+F''$ but not by F' alone nor by F'' alone. Suppose also that the intersection $F'F''$ is $(n-2)$-dimensional. Then we can find an arbitrarily small sphere U and in U two closed subsets f', f'' of F', F'', respectively, and two points p', p'' such that p' and p'' are separated in U by the sum $f'+f''$ but not by f' alone nor by f'' alone.[9]

1. Let K be a 2-dimensional complex, not necessarily homogeneous. By a subharmonic division of K we mean a division of the ground-cells of K into four classes[4] A', B', A'', B'' such that no cell of A' meets a cell of A'' and no cell of B' meets a cell of B''. Thus A' is separated from A'' on K by $B=B'+B''$ and B' is separated from B'' on K by $A=A'+A''$.

A subharmonic division of K resembles the division of a circle k into four quadrants a', b', a'', b'' and it is possible to construct a continuous representation of K on k (or on a part of k) in which each class of the subharmonic division of K is transformed on to the corresponding quadrant of k (or on to a part of it).

Let k be oriented, say in the direction

$$a' \longrightarrow b' \longrightarrow a'' \longrightarrow b'' \longrightarrow a',$$

and let Γ be an oriented 1-cycle drawn on K. The continuous representation of K on k gives a continuous representation of Γ on k. The order (in Brouwer's sense) of this representation of Γ on k we call the order of Γ with respect to the subharmonic division. This order is an additive function of the cycle Γ.

The boundary of any 2-cell of K is a cycle of order 0 and hence every 1-cycle homologous to 0 on K is of order 0. In other words the order of a 1-cycle is invariant under homologies on K.

A theory of order mod 2 invariant under homologies mod 2 follows at once.

2. Let Z be an $(n-2)$-dimensional closed point-set in an n-dimensional Euclidean space R_n. Let Λ be a closed polygon not meeting Z. Then there is a 2-complex K not meeting Z on which Λ is homologous to the sum of a number of arbitrarily small cycles arbitrarily close to Z.

To construct such a complex we first choose a point C and form a singular 2-cell E by joining all points of Λ to C by straight segments.[5] The cell E we subdivide into triangles each of diameter less than ϵ_1 (where ϵ_1 is an arbitrarily small positive number assigned in advance) so obtaining a simplicial complex K_1. The subcomplex of K_1 free from points of Z we denote by K_1^*. I.e., K_1^* consists of all vertices of K_1 not lying on Z and all 1-cells and 2-cells of K_1 which do not meet Z.

From K_1 we now form a cell-complex K_1' of the same structure as K_1 with a skeleton not meeting Z.

(1) We retain K_1^* unchanged.

(2) A vertex P_1 of K_1 lying on Z we replace by a point P_1' within a distance ϵ_1 of P_1 but not lying on Z.

(3) An edge P_1Q_1 of K_1 which meets Z we now replace by a path (i.e., chain of segments) $\widehat{P_1'Q_1'}$ lying entirely within a distance ϵ_1 of P_1Q_1 but not meeting Z.[6]

(4) Let Δ_1 be a simplex of K_1 which meets Z. Its perimeter has already been replaced by a closed polygon Λ_1 not meeting Z but lying within a distance ϵ_1 of Δ_1. We now choose a point C_1 within a distance ϵ_1 of Δ_1 and replace Δ_1 by the singular 2-cell E_1 joining C_1 to Λ_1.

The subcomplex of K_1' free from points of Z is the desired complex K. On it Λ is homologous to the sum of the boundaries of those cells of K_1'

which meet Z. Each of these is a cycle of diameter less than $3\epsilon_1$ lying within a distance $3\epsilon_1$ of Z.[7]

3. Let F be the common boundary of two domains G', G'' in R_n. There may of course be other domains also complementary to F and the boundary of any such domain may or may not be identical with F. Let Z be an $(n-2)$-dimensional cut of F so that

$$F - Z = F' + F''$$

where F' and F'' are open subsets of F without common points.

We shall now prove the following theorems:

I*. There lies on F an arbitrarily small $(n-1)$-dimensional Cantor manifold f of the form

$$f = f' + z + f''$$

where

$$f' \subset F', \quad z \subset Z, \quad f'' \subset F'',$$

no one of these three parts of f being empty.

II*. There lies on Z a point which is the limit of a decreasing sequence of $(n-1)$-dimensional Cantor manifolds of this form.

We first construct a polygon Λ linked with Z. Take a point P of F' and inside a neighborhood of P so small as to exclude points of $F'' + Z$ choose points $P' \subset G'$ and $P'' \subset G''$. Then the segment $P'P''$ does not meet $F'' + Z$. Similarly we choose points $Q' \subset G'$ and $Q'' \subset G''$ such that the segment $Q'Q''$ does not meet $F' + Z$. We join P' to Q' by a path in G' and joint P'' to Q'' by a path in G''. Then combining these paths with the segments $P'P''$ and $Q'Q''$ we get the desired polygon Λ.

Next we construct as in §2 a 2-complex K not meeting Z on which Λ is homologous to the sum of a number of small cycles close to Z. Then we subdivide K so finely that no ground-cell which contains a point of F' meets any ground-cell which contains a point of F''. On the resulting complex we define a harmonic division in which

A' consists of all cells meeting F',

A'' consists of all cells meeting F'',

B' consists of all ground-cells contained entirely in G',

B'' consists of all ground-cells contained in some other domain complementary to F.

The order of Λ with respect to this harmonic division is clearly ± 1. By §1 there must be at least one of the small cycles with an order different from 0. Let Λ_1 be such a cycle: it has a diameter less than $3\epsilon_1$ and so can be enclosed in a sphere U_1 of diameter less than $6\epsilon_1$. Our manifold f will be the common boundary[8] relative to U_1 of two domains in U_1.

The cycle Λ_1 is divided "harmonically" into a finite number of arcs belonging alternately to A and to B. If it is possible in U_1 to join two B-arcs

of Λ_1 without meeting F then we do so and in this way express Λ_1 as the sum of two cycles Λ_1', Λ_1'' lying in U_1 and each simpler than Λ_1 in the sense that it is divided into a smaller number of arcs by the harmonic division. Of the two new cycles one at least has an order different from 0 and we can therefore replace Λ_1 by a simpler cycle. By a finite number of such steps we obtain a cycle lying in U_1 and having an order different from 0 with the added property that the B-arcs belong to different domains complementary to F in U_1.

We can similarly reduce Λ_1 if it is possible in U_1 to join a B-arc to any A-arc other than the two adjacent ones without meeting F. And more precisely suppose that the domain complementary to F in U_1 which contains some particular B-arc β has on its boundary a point belonging to an A-arc α not adjacent to β. Then by an arbitrarily small change we can make α pass through this domain. This change does not disturb the harmonic division of Λ_1 and the cycle can now be reduced as before. We may therefore suppose that each B-arc of Λ_1 belongs to a domain complementary to F in U_1 which has a positive distance from all the arcs of Λ_1 other than that B-arc and the two adjacent A-arcs.

The cycle Λ_1 having an order different from 0 must have arcs of all four classes. We can therefore choose three consecutive arcs α', β, α'' of which α' belongs to A', β belongs to B' or B'' and α'' belongs to A''. The domain g complementary to F in U_1 which contains β has a positive distance from the sum σ of all arcs of Λ_1 other than α', β, α''. Its boundary b relative to U_1 does not meet σ but separates σ from β. The domain complementary to b in U_1 which contains σ has for its boundary relative to U_1 the desired Cantor manifold f. By construction it is the common boundary, relative to U_1, of two domains, one domain containing the arc β and the other domain containing the arc σ. Since it separates σ from β it must cut both α' and α'' and hence contains points of F' and also points of F''. That it also meets Z now follows from the fact that it is a Cantor manifold and therefore connected.

A different method of constructing f from the cycle Λ_1 of order different from 0 will be given in another paper. We observe that f is a cut of the sphere U_1 (this is not required in the enunciation of I* and II*) and that it is part of the boundary, relative to U_1, of one connected part of $U_1 - F$.

The construction of a decreasing sequence $\{f_m\}$ of such $(n-1)$-dimensional Cantor manifolds, each the common boundary of two domains relative to a sphere U_m, will now be sufficiently obvious. This proves Theorem II*. The limit-point of such a sequence we call a Risspunkt of the cut Z. The points of F which are Risspunkte for suitable cuts is like the set of $(n-1)$-dimensional manifold points at least 1-dimensional.

The proof of the theorem enunciated in the introduction is obtained by first replacing $F' + F''$ by a surface $F \subset F' + F''$ which is the common

boundary of a domain $G' \supset P'$ and a domain $G'' \supset P''$. To this F we now apply the same arguments as used above in proving I*.

[1] Kaufmann, "Cantor Manifolds Lying on a Closed Surface," *Proc. Camb. Phil. Soc.*, **30** (1934); and Ursell, "Cantor Manifolds, etc.," Part 2, *Ibid.*, **31**, to appear shortly.

[2] Kaufmann, "Der lokale Dimensionsbegriff," *Math. Ann.*, to appear shortly.

[3] Alexandroff, *Ann. Math.*, 2nd Ser., **30**, p. 178 (1928).

[4] Each ground-cell belongs to at least one of the four classes. If the classes are mutually exclusive we call the division harmonic: from any subharmonic division we can form a harmonic one by depriving B' and B'' of ground-cells belonging to A.

[5] If Λ is simple and $n \geqslant 4$ we can choose C so that E shall be free from self-crossings: but this is not important for our present purpose.

[6] Z being only $(n-2)$-dimensional cannot destroy the connection of an n-dimensional region. Cf. §3, footnote 8.

[7] If Λ is simple and $n \geqslant 5$ there is a 2-cell M bounded by Λ, free from self-crossings and such that the intersection MZ is 0-dimensional at most. The complex K_1' is a first approximation to M. Cf. Ursell, loc. cit.

[8] That the common boundary relative to U_1 of two domains in U_1 is an $(n-1)$-dimensional Cantor manifold may be obtained from the generalized Urysohn Theorem (see Alexandroff, loc. cit., p. 154), by a transformation which turns the whole boundary of U_1 into a single point. It is unimportant whether we use closed or open spherical regions: but the transition from the Urysohn Theorem for space to the corresponding result for a region is slightly simpler when the region is closed.

[9] Thus there is in $F'F''$ an arbitrarily small Vietoris cycle (relative to the boundary of U) which is homologous to 0, in the relative sense of Lefschetz homologies, on f' and also on f'' but not on $f'f''$.

THE INTERSECTION OF ARBITRARY CHAINS AND ITS BOUNDARY

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S. Lefschetz in his "Topology"[1] has defined the intersection $C_p \cdot C_q$ of two arbitrary singular p and q chains, C_p and C_q, on an orientable n-manifold K-L, L a subcomplex of K, $p + q - n > 0$, making the restriction that they do not meet each other's boundaries on K-L.[2] When C_p is simplicial on the complex K, and C_q is simplicial on K^*, the dual of K, the restriction on the chains is removed, and the intersection $C_p \cdot C_q$ is proved to have the boundary

$$F(C_p \cdot C_q) = (-1)^{n-q} F(C_p) \cdot C_q + C_p \cdot F(C_q). \qquad (1)$$

The object of the work announced in the present note is:

(a) To define the intersection and its boundary when no restriction is imposed upon the arbitrary singular chains C_p and C_q.

(b) To extend (1) into a boundary relation for this intersection.

In order to simplify this outline, we shall assume L to be vacuous. Let C_p and C_q be unrestricted chains; let G be the geometric intersection of C_p and C_q; G' the sum of the geometric intersections of $F(C_p)$ and C_q, and C_p and $F(C_q)$; and G'' the geometric intersection of $F(C_p)$ and $F(C_q)$. Then $G \supset G' \supset G''$. If $\mathfrak{N}'$ is an arbitrary neighborhood of G' on K, then $C_p \cdot C_q \bmod \mathfrak{N}'$ is defined by Lefschetz as a locally homologous family (l.h.f.) of cycles mod $\mathfrak{N}'$ about G. We define $C_p \cdot C_q$ as the infinite sequence of approximating chains $\{C_p^\alpha \cdot C_q^{*\alpha}\}$, which mod $\mathfrak{N}'$ become this l.h.f..

It can be proved that the sequence of boundary cycles $\{F(C_p^\alpha \cdot C_q^{*\alpha})\}$ form a l.h.f. about G'. This l.h.f. we define as the boundary, $F(C_p \cdot C_q)$, of $C_p \cdot C_q$.

In the case when C_p, C_q are unrestricted chains, the following expression now has meaning:

$$(-1)^{n-q} F(C_p) \cdot C_q + C_p \cdot F(C_q). \tag{2}$$

Formally it is the same as the right-hand side of (1); but in (1) the intersecting chains and intersections are simplicial and the r.h.s. is a cycle, while in (2) each intersection is a l.h.f. as follows:

$$\{(-1)^{n-q} \Gamma_{p-1}^\alpha \cdot C_q^{*\alpha}\} + \{C_p^\beta \cdot \Gamma_{q-1}^{*\beta}\} \tag{3}$$

$$(\alpha, \beta = 1, 2, 3, \ldots \text{ independently.})$$

The superscripts here refer to complexes K^α and K^β which may either be subdivisions of K or of other complexes in point to point coincidence with K whose mesh approaches zero as α, β become infinite. The chain $\Gamma_{p-1}^\alpha \cdot C_q^{*\alpha}$ is then the intersection of the deform on K^α of $F(C_p)$ with the deform on $K^{*\alpha}$ of C_q, and $C_p^\beta \cdot \Gamma_{q-1}^{*\beta}$ is the intersection of the deform on K^β of C_p with the deform on $K^{*\beta}$ of $F(C_q)$.

By means of the intersection Theory in Topology, it may be shown that given $\epsilon > 0$, then for α, β above a certain N, there exists within ϵ of G'' a chain

$$-X_{s-1}^{\alpha\beta} \longrightarrow F[(-1)^{n-q} \Gamma_{p-1}^\alpha \cdot C_q^{*\alpha} + C_p^\beta \cdot \Gamma_{q-1}^{*\beta}].$$

This chain determines a i.h.f. for a certain chain X_{s-1} and hence there can be given a l.h.f. about G' for a chain naturally described as

$$(-1)^{n-q} F(C_p) \cdot C_q + C_p \cdot F(C_q) + X_{s-1} \tag{4}$$

In particular if the boundaries of C_p, C_q do not intersect $G'' = 0$ and hence $X = 0$. In any case (4) defines the same l.h.f. that defines $F(C_p \cdot C_q)$. Consequently the extension of (1) is: "The l.h.f. $F(C_p \cdot C_q)$ is equivalent to the l.h.f.

$$(-1)^{n-q} F(C_p) \cdot C_q + C_p \cdot F(C_q) + X_{s-1}."$$

When C_p and C_q have non-intersecting boundaries, $X = 0$ and we have (1).

The steps indicated here can equally well be carried out for the more general manifold K-L considered by Lefschetz. It is in this form that the proofs will appear.

Incidentally the work contains a proof of the topological invariance of intersection cycles not dependent on § 3, ch. iv of "Topology." This proof has the advantage of simplicity but has the disadvantage of not giving a "combinatorial" invariance proof free of continuity considerations.

[1] *Amer. Math. Soc. Colloquium Lectures* (1930).

[2] Terms and notations used throughout are as in 1.

THE GENERAL CASE OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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1. Integro-differential equations are important in several connections—one of them being that, as pointed out by E. Hilb,[1] they may arise as limiting forms of differential systems when the number of dependent variables is indefinitely increased. We consider the equation of the form

$$L(y(x)) = V(y(x)), \tag{1}$$

where

$$L(y(x)) \equiv y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \ldots + a_n(x)y(x), \tag{1a}$$

$$V(y(x)) \equiv x^r a(x) + \int_x^\infty w(\tau, x)y(\tau)d\tau \qquad (a(x) \not\equiv 0). \tag{1b}$$

Here the $a_i(x)$ and $a(x)$ are analytic for $|x| \geqq p_0 > 0$ $(x \neq \infty)$ and they are either given by convergent series of the form

$$a_q x^{\frac{q}{p}} + \ldots + a_1 x^{\frac{1}{p}} + a_0 + a_{-1} x^{-\frac{1}{p}} + \ldots \tag{2}$$

(integer $p \geqq 1$; $a(x)$ without positive powers of x)

or, more generally, they are asymptotic in certain regions extending to infinity, to such possibly divergent series. In the latter case the results would relate to correspondingly restricted regions. The function $w(\tau, x)$ will be taken of the form

$$w(\tau, x) = \tau^\alpha x^\beta \sum_{i,j=0}^{\infty} w_{i,j}\, \tau^{-\frac{i}{p}} x^{-\frac{j}{p}} \qquad (w_{0,0} \neq 0), \tag{3}$$

$$[R(\alpha + \beta) + 2 = -d < 0,\ R(\alpha + r) + 2 = -d_1 < 0,$$
$$R(r - r_\lambda) + 1 < 0],^2$$

the series being convergent for $|\tau|, |x| \geqq \rho_0$. Here the r_λ ($\lambda = 1, \ldots n$) depend on L only and they are defined in (4), below.

The special case, when $x = \infty$ is a regular singular point for the equation $L(y) = 0$, has been treated in a recent note by D. V. Jonesco.[3] *In the present paper the operator L is not restricted in any way;* the treatment is based on our recent papers on differential equations which in the sequel will be referred to as T[4] and T_1.[5]

2. To avoid repetition it will be taken for granted that the main existence results of T and T_1 are known to the reader. Let R_1 be any infinite region for which, according to the fundamental existence theorem of T, the equation $L(y) = 0$ possesses a full set of analytic solutions $y_i(x)$ ($i = 1, \ldots n$) such that, in R_1,

$$y_i(x) \sim e^{Q_i(x)} x^{r_i} \sigma_i'(x) \qquad (i = 1, \ldots n). \tag{4}$$

Here the $Q_i(x)$ are polynomials in $x^{\frac{1}{K}}$ (integer $K \geqq 1$) and the $\sigma_i'(x)$ are formal, possibly divergent series of the type designated in T as "σ-series." Let R_2 be a subregion of R_1 such that, as asserted by the theorem of T_1, the equation $L(y(x)) = x^r a(x)$ has a solution $y(x)$, analytic in R_2 and satisfying an asymptotic relation

$$y(x) \sim x^{r_1}\sigma(x) \qquad (\sigma(x) \text{ a } \sigma\text{-series};\ x \text{ in } R_2). \tag{5}$$

Let R be a subregion, extending to infinity and forming part of the above region, R_2, such that

$$RQ_i(x) \geqq 0 \qquad (i = 1, \ldots n;\ x \text{ in } R).^6 \tag{6}$$

The following existence theorem will be stated.

THEOREM. *Suppose that corresponding to a given operator L* (cf. (1a)), *there exists some region R satisfying the above statement in italics. The integro-differential equation $L(y) = V(y)$* (cf. (1b), (3)) *will then possess a solution analytic in R ($|x| \geqq \rho_0$; $|x| \neq \infty$; ρ_0 sufficiently great).*

In proving this theorem we define functions $z_i(x)$, $c_i(x)$ ($i = 0, 1, \ldots$) by the relations

$$L(z_0(x)) = x^r a(x) \equiv c_0(x), \tag{7}$$

$$L(z_i(x)) = c_i(x) \equiv \int_x^\infty w(\tau, x) z_{i-1}(\tau) d\tau \quad (i = 1, 2, \ldots). \tag{7a}$$
(integrations in R).

It can be verified that such functions can actually be defined, on noting that an equation $L(z) = c(x)$, where $c(x)$ is of the general type of $c_0(x)$, has a solution

$$z(x) = \sum_{\lambda=1}^{n} y_{\lambda,1}(x) \int c(x) \bar{y}_{n,\lambda}(x) dx. \tag{8}$$

Here the $y_i(x)\,[= y_{i,1}(x)]$ are functions occurring in (4). Moreover, the $\bar{y}_{i,j}(x)$ $(i,j = 1, \ldots n)$ are the elements of the inverse of the matrix $(y_{i,1}^{(j-1)}(x))$. The involved integrations are according to the integration methods of T.

In view of (4) we have, for x in R and $\lambda = 1, \ldots n$,

$$\bar{y}_{n,\lambda}(x) \sim e^{-Q_\lambda(x)} \quad x^{-r_\lambda}. \quad (\sigma\text{-series}). \tag{9}$$

On taking account of (6) and (9) it follows, by virtue of the Note in (T; p. 202), that *the involved integrations extend to infinity along paths lying in* R.

At each consecutive step using the theorem of T_1, we prove by induction that the $z_i(x)$, $c_i(x)$ $(i \geqq 0)$ are analytic in R $(x \neq \infty\,;\ |x| \geqq \rho_0;\ \rho_0$ sufficiently great) and that, for x in R and $i = 1, 2, \ldots$,

$$z_{i-1}(x) \sim x^{\alpha_i - 1} \quad s_{i-1}(x), \tag{10}$$

$$c_i(x) \sim x^{\beta_i} \quad \sigma_i \quad (x).$$

Here the $s_{i-1}(x)$ and the $\sigma_i(x)$ denote "σ-series" with possibly a few leading terms in the involved power series absent. Furthermore,

$$\alpha_{i-1} = \beta_{i-1} + 1,\ \beta_i = \alpha + \beta + 1 + \alpha_{i-1}\ (i = 1, \ldots;\ \beta_0 = r) \tag{10a}$$

so that

$$R(\alpha_i - \alpha_{i-1}) = R(\alpha + \beta) + 2 = -d < 0\ (i = 1, 2, \ldots). \tag{10b}$$

Thus, in (10)

$$\alpha_i = r + 1 - id, \quad \beta_i = r - id \qquad (i = 0, 1, \ldots). \tag{11}$$

In the σ-series occurring in (4) and (9) $\log x$ may appear to at most the k-th power $(k \geqq 0)$. Let

$$\varphi(x) = |\log^k(x)|. \tag{12}$$

The relations (4) and (9) are then seen to imply that, for x in R,

$$|y_{\lambda,1}(x)| < |e^{Q_\lambda(x)} x^{r_\lambda}|\, \sigma_1\, \varphi(x), \tag{13}$$

$$|\bar{y}_{n,\lambda}(x)| < |e^{Q_\lambda(x)} x^{-r_\lambda}|\, \sigma_1 \varphi(x) \qquad (\lambda = 1, \ldots n). \tag{13a}$$

It will be established that with sufficiently great z and f, independent of x and i, we have

$$|z_v(x)| < z f^v \varphi^{2v+2}(x)\, |x|^{Rr+1-vd}, \tag{14}$$

$$|c_v(x)| < z f^v\, \varphi^{2v}(x)\, |x|^{Rr-vd} \qquad (v = 0, 1, \ldots), \tag{14a}$$

for x in $R(|x| \geqq \rho > 0)$. These inequalities are verified to hold for $v = 0, 1$ (when z and f are suitably chosen). Suppose (14), (14a) hold for $v = 0, 1, \ldots, i-1$ $(i - 1 \geqq 1)$. Then, on using (7a), it follows that

$$| c_i(x) | < w \int_x^\infty | \tau^\alpha | \, | x^\beta | \, | zf^{i-1}\varphi^{2i}(\tau) | \, | \tau |^{Rr+1-id+d} \, | d\tau |$$
$$< q_i z f^{i-1}\varphi^{2i}(x) \, | x |^{Rr-id} \qquad (q_i = a_i w). \tag{15}$$

In consequence of (15), (7a) and of a formula of type (8) we get, by virtue of (13) and (13a),

$$| z_i(x) | < \sum_{\lambda=1}^{n} | e^{Q\lambda(x)} x^{r_\lambda} | \, | \varphi(x)\sigma_1^2 \int_x^\infty z q_i f^{i-1} \varphi^{2i+1}(\tau) \cdot$$
$$| \tau |^{R(r-r_\lambda)-id} \cdot | e^{-Q\lambda(\tau)} | \, | d\tau | < q_i^1 z q_i f^{i-1} \varphi^{2i+2}(x) \, | x |^{Rr+1-id}; \tag{16}$$

here $q_i' = n\sigma_1^2 a_i'$.

The constants a_i, a_i', occurring in (15) and (16) are from the following inequalities

$$\int_x^\infty | \tau |^{R(\alpha+r)+1+d-id} \varphi^{2i}(\tau) \, | d\tau | < | x |^{R(\alpha+r)+2+d-id} \varphi^{2i}(x) \, a_i, \tag{17}$$

$$\int_x^\infty | \tau |^{R(r-r_\lambda)-id} \varphi^{2i+1}(\tau) e^{-RQ\lambda(\tau)} \, | d\tau |$$
$$< | x |^{R(r-r_\lambda)-id+1} \varphi^{2i+1}(x) e^{-RQ\lambda(x)} a_i' \qquad (\lambda = 1, \ldots n) \tag{17a}$$

and they can be chosen so that

$$q_i, q_i' < f^{1/2} \qquad (i = 2, 3, \ldots), \tag{18}$$

where f is independent of i. We select f, as is possible, $\geqq 1$ so that (18) holds and so that inequalities (14), (14a) are valid for $v = 0, 1$ Consideration of (15) and (16) will then make it clear that the inequalities (14), (14a) are true for $v = 0, 1, \ldots$.

In view of (14) it is possible to assert that there exists a positive number ρ_0 such that the series

$$z(x) = \sum_{v=0}^{\infty} z_v(x) \tag{19}$$

is absolutely and uniformly convergent in R, for $| x | \geqq \rho_0$. This series, accordingly, represents a function analytic in the stated region (for $x \neq \infty$). Moreover, by virtue of satisfied convergence conditions and in consequence of (7) and (7a) the function (19) is seen to be a solution of the given integro-differential equation.

[1] E. Hilb, "Zur Theorie der linearen Integrodifferentialgleichungen," *Math. Annal.*, **77**, 514–535 (1915).

[2] Ra = real part of a.

[3] D. V. Jonesco, "Généralisation d'une équation de M. E. Goursat," *Compt. Rend.*, **2**, 666–668 (October, 1933).

[4] W. J. Trjitzinsky, "Analytic Theory of Linear Differential Equations," *Acta Math.*, **62–1**, 2, 167–226 (1934).

[5] W. J. Trjitzinsky, "The General Case of Non-Homogeneous Linear Differential Equations," *Proc. Nat. Acad. Sci.*, **19**, 687–690 (1933).

[6] Such a region exists not in all instances.

[7] An analogous result holds when $a(x) \equiv 0$, unless zero is the only solution.

CONFORMAL TENSOR INVARIANTS

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The conformal geometry of Riemann spaces was developed by T. Y. Thomas[1] as the invariant theory of a relative quadratic differential form of weight $\left(-\frac{2}{n}\right)$. He devised a conformal tensor calculus and obtained a complete set[2] of conformal tensor invariants. The present paper offers an alternative complete set of invariants derived by the use of a general method of completing conformal tensors. In particular a new definition of conformal covariant derivative arises

1. It will be the convention to denote affine indices $(1, 2, \ldots, n)$ by capital Greek letters, projective indices $(0, 1, \ldots, n)$ by small Greek letters, conformal indices $(0, 1, \ldots, n, \infty)$ by Roman letters. $G_{\Phi\Psi}$ is the fundamental symmetric tensor of weight $-\frac{2}{n}$, $\mathfrak{C}_{ij}$ the corresponding symmetric conformal tensor defined by $\mathfrak{C}_{\Phi\Psi} = G_{\Phi\Psi}$, $\mathfrak{C}_{0\infty} = \mathfrak{C}_{\infty 0} = -1$, other components zero.

Associated with the coördinate transformation

$$\bar{x}^{\Delta} = \bar{x}^{\Delta}(x^1 \ldots x^n) \tag{1.1}$$

is the matrix

$$u^i_j: \quad \begin{Bmatrix} & \bar{\Psi}_1 & \cdots & \bar{\Psi}_n & \frac{1}{2}\bar{\Psi}^{\Sigma}\bar{\Psi}_{\Sigma} \\ 0 & \frac{\partial x^1}{\partial \bar{x}^1} & \cdots & \frac{\partial x^1}{\partial \bar{x}^n} & \bar{\Psi}^{\Sigma}\frac{\partial x^1}{\partial \bar{x}^{\Sigma}} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & \frac{\partial x^n}{\partial \bar{x}^1} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} & \bar{\Psi}^{\Sigma}\frac{\partial x^n}{\partial \bar{x}^{\Sigma}} \\ 0 & 0 & \cdots & 0 & (x\bar{x})^{\frac{2}{n}} \end{Bmatrix} \tag{1.2}$$

$$\bar{\Psi}_{\Sigma} = \frac{\partial \log (x\bar{x})}{\partial \bar{x}^{\Sigma}} \qquad \bar{\Psi}^{\Delta} = \bar{G}^{\Delta\Sigma}\bar{\Psi}_{\Sigma}.$$

It is used as the transformation matrix of the conformal indices of tensors while the submatrices $\|u^{\alpha}_{\beta}\|$, $\|u^{\Delta}_{\Sigma}\|$ serve the same purpose for projective and

affine indices, respectively. By the use of an auxiliary variable x°, $\|u^\alpha_\beta\|$ may be viewed as the Jacobian of (1.1) together with $\bar{x}^\circ = x^\circ + \log(x\bar{x})$. In the following, all functions are assumed independent of x° but the process of differentiating with respect to it will be used to simplify the formalism.

${}^\circ\Gamma^i_{j\alpha}$ is the associated conformal connection[4] with the law of transformation

$$ {}^\circ\bar{\Gamma}^i_{j\alpha} = \left({}^\circ\Gamma^r_{s\beta}\, u^s_j\, u^\beta_\alpha + \frac{\partial u^r_j}{\partial \bar{x}^\alpha} \right) v^i_r, \qquad u^i_j\, v^j_\kappa = \delta^i_k. $$

Finally ${}^\circ B^i_{j\alpha\beta}$ is the incomplete conformal curvature tensor.[5]

2. In Riemannian geometry it will be recalled a complete set of tensor invariants is formed by the fundamental tensor together with the curvature tensor and its successive covariant derivatives. It is indicated therefore to seek an analogous system for the conformal case. Now from a completely conformal tensor of weight N, say $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}$, a projective covariant derivative[6] may be constructed by the use of ${}^\circ\Gamma^i_{j\alpha}$ to give an incomplete conformal tensor

$$ T^{i_1 \ldots i_r}_{j_1 \ldots j_s;\,\alpha} = \frac{\partial T^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial x^\alpha} + T^{l i_2 \ldots i_r}_{j_1 \ldots j_s}\, {}^\circ\Gamma^{i_1}_{l\alpha} \ldots - T^{i_1 \ldots i_r}_{j_1 \ldots j_{s-1} l}\, {}^\circ\Gamma^l_{j_s\alpha} - N T^{i_1 \ldots i_r}_{j_1 \ldots j_s}\, {}^\circ\Gamma^l_{l\alpha}. \tag{2.1} $$

But the operation cannot be repeated because the new tensor is not complete. And upon the incomplete curvature tensor it cannot be performed at all. Thus if further progress is to be made in this direction, some invariant method must be found to complete projective indices.[7]

In the case of a contravariant index the simplest procedure is obvious. If $T^{\alpha}_{\cdot\, j\, \cdot}$ is a tensor with a projective contravariant index α it may be completed by the invariant definition $T^{\infty}_{\cdot\, j\, \cdot} = 0$.

In the case of a covariant index, the simplest procedure is brought to light most readily by a geometrical discussion. The totality of all conformal contravariant vectors X^i at a point x may be viewed as an $n+1$-dimensional projective space P_{n+1} in which the components of the vectors are homogeneous coördinates and in which the tensor $\mathfrak{C}_{ij}$ determines a quadric

$$ \mathfrak{C}_{ij} X^i\, X^j = 0. \tag{2.2} $$

The changes of coördinates given by (1.2) leave invariant the point $(1, 0, \ldots, 0)$ and the n-plane $X^\infty = 0$. A covariant projective vector, say φ_α, represents an $(n-1)$ plane in $X^\infty = 0$. The simplest[8] configuration in P_{n+1} determined by this $(n-1)$ plane is the family of all n-planes containing it, that is to say, a special null system with the $(n-1)$ plane as singular element. A null system in P_{n+1} is represented analytically by a

conformal skew-symmetric covariant tensor of the second order. In particular, the null system determined by φ_α is

$$\varphi_{ij} = A^{\alpha}_{ij}\, \varphi_\alpha$$

where

$$A^{\alpha}_{ij} = \delta^{\alpha\infty}_{ij\infty} \equiv \delta^{\alpha}_{i}\, \delta^{\infty}_{j} - \delta^{\alpha}_{j}\, \delta^{\infty}_{i}. \tag{2.3}$$

A^{α}_{ij} is a numerical tensor of weight $-\frac{2}{n}$. Hence if $T^{\cdots}_{\cdots\alpha\cdots}$ is any incomplete-

conformal tensor of weight N the index α may be replaced by a skew-symmetric pair of conformal indices giving rise to a tensor of weight $N - \frac{2}{n}$,

$$T^{\cdots}_{\cdots ij\cdots} = A^{\alpha}_{ij}\; T^{\cdots}_{\cdots\alpha\cdots} \tag{2.4}$$

or more explicitly $T^{\cdots}_{\cdots\alpha\infty\cdots} = -T^{\cdots}_{\cdots\infty\alpha\cdots} = T^{\cdots}_{\cdots\alpha\cdots}$, others zero. Needless to say, this operation can be applied to any number of covariant projective indices simultaneously, thus furnishing, together with the convention for contravariant indices, a process for completing any tensor. It should be remarked that the completed tensor is uniquely determined by the given incomplete one and conversely.

The projective covariant derivative completed by the above device

$$T^{i_1\ldots i_r}_{j_1\ldots j_s/kl} = A^{\alpha}_{xl}\; T^{i_1\ldots i_r}_{j_1\ldots j_s;\,\alpha} \tag{2.5}$$

will be called the conformal covariant derivative.

Finally it should be noted that the completed curvature tensor

$$B^{i}_{jklk_0l_0} = A^{\alpha}_{kl}\, A^{\beta}_{k_0l_0}\, {}^{\circ}B^{i}_{j\alpha\beta},$$

has the components

$$B^{i}_{j\alpha\infty\beta\infty} = B^{i}_{j\infty\alpha\infty\beta} = -B^{i}_{j\infty\alpha\beta\infty} = -B^{i}_{j\alpha\infty\infty\beta} = {}^{\circ}B^{i}_{j\alpha\beta}, \tag{2.6}$$

other components zero.

3. The completed curvature tensor together with its successive conformal derivatives forms a sequence of tensor invariants

$$B^{i}_{jklk_0l_0},\; B^{i}_{jklk_0l_0/k_1l_1},\; \ldots,\; B^{i}_{jklk_0l_0/\ldots/k_rl_r},\; \ldots \tag{3.1}$$

of the respective weights $\frac{-4}{n}, \frac{-6}{n}, \ldots, -\frac{2r+4}{n}, \ldots$

THEOREM. The sequence (3.1) together with $G_{\Phi\Psi}$ forms a complete set of tensor invariants for the conformal geometry of Riemannian spaces.

To prove this consider the differential equations[9]

$$\frac{\partial u^i_j}{\partial \bar{x}^\alpha} = {}^\circ\bar{\Gamma}^r_{j\alpha}\, u^i_r - {}^\circ\Gamma^i_{k\beta}\, u^k_j\, u^\beta_\alpha \tag{3.2}$$

$$\frac{\partial x^\alpha}{\partial \bar{x}^\beta} = u^\alpha_\beta$$

together with the finite relations

$$\begin{cases} \bar{G}_{\Phi\Psi} = \left| u^\Lambda_\Pi \right|^{-\frac{2}{n}} G_{\Delta\Omega}\, u^\Delta_\Phi\, u^\Omega_\Psi \\ u^\Delta_\infty = \bar{G}^{\Phi\Psi}\, u^\Delta_\Phi\, u^0_\Psi \\ u^0_\infty = {}^1/_2 \bar{G}^{\Phi\Psi}\, u^0_\Phi\, u^0_\Psi \quad u^\infty_\infty = \left| u^\Lambda_\Pi \right|^{2/n} \\ u^i_0 = \delta^i_0 \quad u^\infty_\Delta = 0 \end{cases} \tag{3.3}$$

$$B^i_{jklk_0l_0}\, u^p_i = B^p_{qrsr_0s_0}\, u^q_j\, u^r_k\, u^s_l\, u^{r_0}_{k_0}\, u^{s_0}_{l_0}. \tag{3.4}$$

Because of (2.6) equations (3.4) are the conditions of integrability of (3.2). Also, by virtue of (2.1), (2.5), the equations obtained by differentiating (3.4) successively and eliminating by means of (3.2) are algebraically equivalent to the laws of transformation $B \longrightarrow \bar{B}$ of the successive conformal derivatives (3.1). Differentiation of (3.3) and elimination by (3.2) leads to identities. The theorem then follows by applying the well-known lemma[10] on the existence of solutions of a mixed system of total differential equations to the system (3.2, 3.3, 3.4).

4. There is a device which can be used to complete a covariant projective index which replaces it by a single conformal index instead of a skew-symmetric pair. It is the substitution (cf. 2.4)

$$T_{.\alpha.} \longrightarrow T_{.i.} \equiv T_{.ij.}\, X^j \tag{4.1}$$

where X^j is a set of $n + 2$ independent variables transforming under (1.1) as a contravariant conformal vector. The completion of r covariant projective indices leads to forms of degree r in X. The use of tensors whose components are functions of an independent conformal vector is the conformal analogue of the use of affine tensors whose components are functions of the differentials dx, a frequent occurrence in differential geometry. Indeed it is convenient and suggestive to call the variables X^i conformal differentials.

The projective covariant derivative of a conformal tensor involving the variables X is obtained from the formula (2.1) by the addition of the term

$$-X^k \frac{\partial T^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial X^l}\, {}^\circ\Gamma^l_{k\alpha} \tag{4.2}$$

and may be completed to a conformal tensor by the above device.

The application of this process of completion and covariant differentiation to ${}^\circ B^i_{j\alpha\beta}$ gives rise to the sequence of tensor invariants

$$B^i_{jkk_0}(X),\ B^i_{jkk_0/k_1}(X),\ \ldots,\ B^i_{jkk_0/k_1/\ldots/k_r}(X),\ \ldots \qquad (4.3)$$

whose components are respectively forms of degree $2, 3, \ldots, r+2, \ldots$ in X. This sequence as well as (3.1) forms a complete set of invariants, the relationship between them being given by

$$B^i_{jkk_0/k_1/\ldots/k_r} = B^i_{jkl k_0 l_0/k_1 l_1/\ldots/k_r l_r} X^l X^{l_0} \ldots X^{l_r} \quad r = 0, 1, \ldots \qquad (4.4)$$

[1] "Invariants of Relative Quadratic Differential Forms," these PROCEEDINGS, **11**, 722–725 (1925). "On Conformal Geometry" (referred to hereafter as C.G.), *Ibid.*, **12**, 352–359 (1926). "Conformal Tensors (First and Second Note)," *Ibid.*, **18**, 103–112, 189–193 (1932). The results of these papers may also be found in his book *Differential Invariants of Generalized Spaces*, Cambridge, 1934 (referred to hereafter as D.I.), 66–83, 208–212.

[2] That is, a set in terms of which there may be stated algebraic necessary and sufficient conditions for the conformal equivalence of two Riemannian spaces.

[3] C.G. p. 356. D.I. p. 69.

[4] C.G. p. 357. D.I. p. 70.

[5] C.G. p. 357. D.I. p. 72. An incomplete conformal tensor is one with one or more projective indices.

[6] O. Veblen, "Conformal Tensors and Connections," these PROCEEDINGS, **14**, 735–745, § 8 (1928); T. Y. Thomas, "Conformal Tensors," these PROCEEDINGS, **18**, 103–112 (2.3) (1932).

[7] O. Veblen was first to suggest such a process, loc. cit. §§ 7, 8.

[8] The only apparently simpler alternative is the n-plane tangent to (2.2) and containing the $(n-1)$-plane. The coördinates of this n-plane are $(\varphi_\alpha, \varphi_\infty)$, a conformal vector, where φ_∞ is defined by the invariant condition $\mathfrak{C}^{ij}\,\varphi_i\,\varphi_j = 0$. This method of completion because of its non-linearity fails to apply to higher order incomplete tensors.

[9] Cf. D.I. p. 211.

[10] O. Veblen and J. M. Thomas, "Projective Invariants of Affine Geometry of Paths," *Ann. Math.*, **27**, 279–296, § 7 (1926). D.I. p. 202, § 84.

INVOLUTORIAL SIMPLE ALGEBRAS AND REAL RIEMANN MATRICES

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1. *Involutorial Simple Algebras.*—Let $\mathfrak{A}$ be a simple algebra over a field $\mathfrak{F}$ of characteristic not two and let there be a correspondence

$$J: \qquad a \longleftrightarrow a^J \qquad (a, a^J \text{ in } \mathfrak{A})$$

of $\mathfrak{A}$ such that

$$(a+b)^J = a^J + b^J,\ (ab)^J = b^J a^J,\ (a^J)^J = a,\lambda^J = \lambda$$

for every a and b of $\mathfrak{A}$ and λ of $\mathfrak{F}$. Then J is an *involution* of $\mathfrak{A}$ and $\mathfrak{A}$ is J-

involutorial. If $a = a^J$ then a is J-symmetric and if $a = -a^J$ then a is J-skew. Every a of $\mathfrak{A}$ is the sum of a J-symmetric and a J-skew quantity of $\mathfrak{A}$. In particular the centrum of $\mathfrak{A}$ is either a field $\mathfrak{K}_0 = \mathfrak{K}$ of J-symmetric quantities or $\mathfrak{K} = \mathfrak{K}_0(q)$, $q^2 = \mu$ in $\mathfrak{K}_0$, $q^J = -q$. If T is also an involution of $\mathfrak{A}$ such that $k_0{}^T = k_0$, $q^T = -q$ for every k_0 of $\mathfrak{K}_0$ then we call T a *J-centrum preserving involution* of $\mathfrak{A}$. I have proved

THEOREM 1. *A self correspondence* T *of* $\mathfrak{A}$ *is a* J*-centrum preserving involution of* $\mathfrak{A}$ *if and only if there exists a regular quantity* $\mathrm{p} = \pm \mathrm{p}^{\mathrm{J}}$ *in* $\mathfrak{A}$ *such that*

$$a^{\mathrm{T}} = p^{-1}a^{\mathrm{J}}p. \tag{1}$$

COROLLARY. *If* Z *is a maximal sub-field of* $\mathfrak{A}$ *and* $\mathrm{z}^{\mathrm{T}} = \mathrm{z}^{\mathrm{J}}$ *for every* z *of* Z *then* (1) *holds with* p *in* Z.

Let $\mathfrak{A} = \mathfrak{M} \times \mathfrak{D}$ where $\mathfrak{M}$ is a total matric algebra with basis (e_{ij}) $\mathfrak{D}$ is a division algebra. Then $\mathfrak{M}$ is T-involutorial with $\mathrm{e_{ij}}^{\mathrm{T}} = \mathrm{e_{ji}}$. We have

THEOREM 2. *If* $\mathfrak{A} = \mathfrak{M} \times \mathfrak{D}$ *is* J*-involutorial there exists a* J*-centrum-preserving involution* T *of* $\mathfrak{A}$ *such that* $\mathfrak{D}$ *is* T*-involutorial and* $\mathrm{e_{ij}}^{\mathrm{T}} = \mathrm{e_{ji}}$.

THEOREM 3. *Let* $\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$ *be a* J*-involutorial division algebra, where* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *have relatively prime degrees over* $\mathfrak{K}$. *Then there exists a* T *such that* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *are each* T*-involutorial.*

Every $\mathfrak{D}$ is similar to a crossed product (Z,a) where Z is a galois field over $\mathfrak{K}$ and $a = (a_{\mathrm{G,H}})$ with $a_{\mathrm{G,H}}$ in Z and G,H ranging over all automorphisms of Z. If $\mathfrak{K} = \mathfrak{K}_0$ then we have

THEOREM 4. *Algebra* $\mathfrak{D} \neq \mathfrak{K}_0$ *is* J*-involutorial over* $\mathfrak{K} = \mathfrak{K}_0$ *if and only if* $\mathfrak{D}$ *has exponent two.*

If $\mathfrak{K} = \mathfrak{K}_0(q) \neq \mathfrak{K}_0$ then we may in general take $Z = Z_0 \times \mathfrak{K}$ where Z_0 is galois over $\mathfrak{K}_0$ and write $a_{\mathrm{G,H}} = a_{\mathrm{G,H1}} + a_{\mathrm{G,H2}}q$, $a_{\mathrm{G,H1}}$ and $a_{\mathrm{G,H2}}$ in Z_0. Then $b_{\mathrm{G,H}} = a_{\mathrm{G,H1}}{}^2 - a_{\mathrm{G,H2}}{}^2\mu$ are in Z_0 and $\beta = (Z_0, b_{\mathrm{G,H}})$ is a crossed product over $\mathfrak{K}_0$. I have proved

THEOREM 5. *Algebra* $\mathfrak{A} = (\mathrm{Z,a})$ *is* J*-involutorial if and only if* $\mathfrak{B}$ *is a total matric algebra.*

THEOREM 6. *Let* Z_0 *be cyclic so that* $\mathfrak{A} = (\mathrm{Z,S},\gamma)$, $\gamma = \gamma_1 + \gamma_2\mathrm{q}$ *with* γ_1 *and* γ_2 *in* $\mathfrak{K}_0$. *Then* $\mathfrak{A}$ *is* J*-involutorial if and only if* $\gamma_1{}^2 - \gamma_2{}^2\,\mu$ *is the norm over* $\mathfrak{K}_0$ *of a quantity* d *in* Z_0.

We may consider in particular the case where $\mathfrak{F}$ is an algebraic number field and show that every $\mathfrak{D}$ over $\mathfrak{K}_0$ has degree 1 or 2 and every $\mathfrak{D}$ over $\mathfrak{K} = \mathfrak{K}_0(q)$ is a cyclic algebra of Theorem 6. In particular these results may be applied to the case of the multiplication algebras of pure Riemann matrices where $\mathfrak{K}_0$ is total real, $q^2 = \mu$ is total negative and we show that z_0 is total real, d must be total positive.

2. *Weyl Matrices.*—Let Γ be a J-involutorial field containing the least algebraically closed extension Λ of a field $\mathfrak{F}$ of characteristic not two and

assume that α^J is in $\mathfrak{F}$ for every α of $\mathfrak{F}$. Let also Γ_0 be the set of all J-symmetric quantities of Γ and similarly for $\mathfrak{F}_0$ and Λ_0. The set $\mathfrak{M}$ of all p-rowed square matrices σ with elements in Γ is a J-involutorial simple algebra with J given by

$$\sigma = (\sigma_{ij}),\ \sigma^J = (\rho_{ij}),\ \rho_{ij} = \sigma^J_{ji}\ (i,j = 1, \ldots, p).$$

If τ is a J-symmetric matrix of $\mathfrak{M}$ and is totally regular in Λ, and if $\tau = RC$ where $C^J = \pm C$ has elements in $\mathfrak{F}$ then we call R a Weyl[1] matrix over $\mathfrak{F}$. The set $\mathfrak{A}$ of all square matrices A with elements in $\mathfrak{F}$ such that $AR = RA$ is a linear associative algebra over $\mathfrak{F}$ called the *multiplication algebra* of R. Two Weyl matrices R and R_0 are said to be isomorphic if $R_0 = U^{-1}RU$ and associated if $R_0 = WRU$ where W and U have elements in $\mathfrak{F}$ and are non-singular. Isomorphic Weyl matrices have equivalent multiplication algebras and the study of $\mathfrak{A}$ is easily reduced to the case where $\mathfrak{A}$ is a division algebra. If $\Lambda = \Lambda_0$ then it is shown that $\mathfrak{D} = \mathfrak{K}_0$ is a J-symmetric field. If $\Lambda \neq \Lambda_0$ then $\Lambda = \Lambda_0(i)$, $i^2 = -1$ and Λ_0 is a real closed field. In this case we assume that Γ is algebraically closed and hence $\Gamma = \Gamma_0(i)$ where Γ_0 is real closed, $i^2 = -1$. Then τ is a Hermitian matrix and, in order to compose irreducible Weyl matrices, it is desirable to assume that τ is positive definite. This assumption is made and we prove that $\mathfrak{K}_0$ is total real, the field Z_1 of Theorem 5 always exists and is total real, and

THEOREM 7. *Let $\mathfrak{A}$ be an algebra of Theorem 5 so that there exist solutions d_G in Z_0 of*

$$b_{G,H} = (d_G^H d_H)^{-1} d_{GH}. \tag{2}$$

Then if $\mathfrak{A}$ is the multiplication algebra of a Weyl matrix the equations (2) must have total positive solutions d_G.

THEOREM 8. *Let $\mathfrak{A} = (Z,a)$ over $\mathfrak{K} = \mathfrak{K}_0$ be defined by a total real galois field Z and be the multiplication algebra of R. Then*

$$(a_{G,H})^2 = (d_G^H d_H)^{-1} d_{GH} \tag{3}$$

must have total positive solutions d_G in Z.

THEOREM 9. *Let $\mathfrak{D}$ over $\mathfrak{K}_0$ be the multiplication algebra of an irreducible Weyl matrix and with no maximal symmetric sub-field. Then $\mathfrak{D}$ is similar to a crossed product $\mathfrak{A} \times Q$ where $\mathfrak{A}$ is an algebra of Theorem 8 and Q is the algebra of real quaternions.*

The above theorems give necessary conditions that $\mathfrak{D}$ be the multiplication algebra of an irreducible R and it is believed that they are also sufficient, in general.

3. *The $\mathfrak{F}$-Algebra of a Weyl Matrix.*—The set $\mathfrak{B}$ of all matrices A with elements in $\mathfrak{F}$ such that $AR = RB$, where B has elements in $\mathfrak{F}$, is an algebra over $\mathfrak{F}$ called the $\mathfrak{F}$-algebra of R. Algebra $\mathfrak{B}$ contains the multiplication

algebra $\mathfrak{A}$ of R and R is called a *central Weyl matrix* if $\mathfrak{B} = \mathfrak{A}$. Associated Weyl matrices have equivalent $\mathfrak{F}$-algebras.

If $\mathfrak{F} = \mathfrak{F}_0$ and $\Gamma = \Gamma_0(\rho)$ so that $\tau = \tau_1 + \tau_2\rho$ we assume that one of τ_1 and τ_2 is non-singular. We then prove

THEOREM 10. *Every Weyl matrix* R *is associated with a central Weyl matrix* R_0 *such that* $\tau = RC = R_0C_0$. *If* $\mathfrak{F} = \mathfrak{F}_0$ *and* $\tau = \tau_1$ *then we may take* $C_0 = C_0'$.

THEOREM 11. *If* R_1 *is isomorphic to a central Weyl matrix* R_0 *then* R_1 *is a central Weyl matrix.*

THEOREM 12. *Every reducible central Weyl matrix is isomorphic to*[2]

$$R_0 = [R_1 \ldots R_t]$$

which has zero matrices off the diagonal and irreducible central Weyl matrices R_i *as diagonal blocks.*

These results evidently complete the theory of $\mathfrak{F}$-algebras by reducing the problem to the previous case of the multiplication algebras of irreducible Weyl matrices.

4. *Real Riemann Matrices.*—Let Γ_0 be a real closed field, $\Gamma = \Gamma_0(i)$, $i^2 = -1$, and let $\mathfrak{F}$ be a sub-field of Γ_0. If ω is a Riemann matrix over $\mathfrak{F}$ the set of all matrices A such that $\alpha\omega = \omega A$ forms a linear associative algebra, the multiplication algebra of ω. We let $\gamma = i\omega c\bar{\omega}'$ where c is a principal matrix of ω so that if

$$\Omega = \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}, \quad E = [I_p - I_p], \quad \Phi = iE\Omega c\bar{\Omega}' = [\gamma, \gamma].$$

then Φ is positive definite Hermitian. Then

$$R_\omega = i\Omega^{-1}E\Omega, \quad \tau = RC = \Omega^{-1}\Phi(\bar{\Omega}')^{-1}$$

is positive definite real symmetric and R_1 is a matrix of Weyl. The multiplication algebras of R_ω and ω coincide.

If $R_1 = i\Omega_0^{-1}E\Omega_0$ and ω is pure then ω_0 is isomorphic to either ω or $\bar{\omega}$. I have proved

THEOREM 13. *Let* ω *be a pure Riemann matrix not isomorphic to* $\bar{\omega}$ *and let* $R_\omega = i\Omega^{-1}E\Omega = i\Omega_0^{-1}E\Omega_0 = R_{\omega 0}$. *Then* R_ω *is a central Weyl matrix whose multiplication algebra is that of* ω *and* $\omega_0 = \mu\omega$.

THEOREM 14. *Let* $\omega = \nu^{-1}\bar{\omega}N^{-1}$ *be a pure Riemann matrix with multiplication algebra* $\mathfrak{D}$ *and Weyl matrix* $R_\omega = i\Omega^{-1}E\Omega$. *Then the* $\mathfrak{F}$*-algebra* $\mathfrak{B}$ *of* R *is the algebra*

$$\mathfrak{B} = \mathfrak{D} + \mathfrak{D}N$$

with $NR = -RN$, N^2 *in* $\mathfrak{D}$.

A Riemann matrix ω is real if $\nu\omega N = \bar{\omega}$ with $N^2 = I_{2p}$. We have

THEOREM 15. *A pure Riemann matrix ω with Weyl matrix $R_\omega = i\Omega^{-1}E\Omega$ is real if and only if the $\mathfrak{F}$-algebra $\mathfrak{B}$ of R_ω is not a division algebra.*

THEOREM 16. *Every real Riemann matrix is isomorphic to both*

$$(\tau, iI_p), \quad (R, iI_p)$$

where $\tau = RC$ is a positive definite real symmetric matrix with $C' = C$ and central Weyl matrix R.

THEOREM 17. *Let $\omega = (R, iI_p)$ where R is an irreducible central Weyl matrix with multiplication algebra $\mathfrak{D}$ and hence ω has a sub-algebra $\mathfrak{D}_0$, of multiplications*

$$A\omega = \omega a_A, \; a_A = [A,A] \quad (A \text{ in } \mathfrak{D}),$$

equivalent to $\mathfrak{D}$. Then the multiplication algebra $\mathfrak{A}$ of ω is $\mathfrak{D}_0$ if and only if R and R^{-1} are not associated. Moreover if $R^{-1} = -z_2^{-1}Rz$, is associated with R then $\mathfrak{A} = \mathfrak{D}_0 + \mathfrak{D}_0 g$,

$$g = \begin{pmatrix} o & z_1 \\ z_2 & o \end{pmatrix}$$

with g^2 and gfg^{-1} in $\mathfrak{D}_0$ for every f in $\mathfrak{D}_0$.

In fact it is easily shown that $\mathfrak{D}_0$ is the set of all multiplications b of ω such that $EbE^{-1} = b$ and $\mathfrak{D}_0 g$ is the set of all b's such that $EbE^{-1} = -b$. We also have

THEOREM 18. *Let $\omega = (R, iI_p)$ as in Theorem 17 and $\omega_0 = (S, iI_p)$ where S is any Weyl matrix. Then ω and ω_0 are isomorphic if and only if S is associated with either R or R^{-1}.*

We have therefore proved that if $\omega = (R, iI_p)$ with R an irreducible central Weyl matrix then ω cannot be isomorphic to $\omega_0 = (S, iI_p)$ where the $\mathfrak{F}$-algebra of S is not a division algebra. We may therefore make the

DEFINITION. *A real Riemann matrix $\omega = (R, iI_p)$ will be called irreducible if the $\mathfrak{F}$-algebra of R is a division algebra.*

We then have

THEOREM 19. *Every real Riemann matrix is isomorphic to*

$$[\omega_1, \ldots, \omega_t]$$

where the ω_j are irreducible real Riemann matrices and in fact

$$\omega = (R, iI_p), \; R = [R_1, \ldots, R_t), \; \omega_j = (R_j, iI_p),$$

if R is the reducible central Weyl matrix in ω.

THEOREM 20. *An irreducible real Riemann matrix ω is impure if and only if it is isomorphic to*

$$[\omega_1, \bar{\omega}_1]$$

where ω_1 is pure and is not a real Riemann matrix.

We have therefore obtained a complete reduction of any impure real Riemann matrix to pure components and have given the structure of a pure real Riemann matrix in Theorem 17.

[1] H. Weyl considered such matrices in his paper in the October, 1934, number of the *Ann. Math.* and wrote the transposed form $\tau = CR$. It seems more natural to write $\tau = RC$ in view of later applications.

[2] We use this notation for a matrix with zero blocks off the diagonal throughout the report.

ERRATA

In the paper by G. H. Parker on "Acetyl Choline and Chromatophores," these PROCEEDINGS, **20**, 596–599, the following corrections should be noted:

Page 599, Line 10, For *dispersion* read *concentration.*

Page 599, Line 11, For *darkening* read *blanching.*

INDEX

Abelian groups (van Kampen), 434.
Absorption spectrum, ultra-violet (Badger and Barton), 166.
Accuracy of least squares (Sterne), 565, 601.
Acetylc holine (Parker), 596.
Achromobacter ichthyodermis, N. Sp. (Wells and ZoBell), 123.
Acid, formic (Pauling and Brockway), 336.
Activator, crustacean chromatophore (Kropp and Crozier), 453.
Active immunization in typhus fever (Zinsser and Castaneda), 9.
Age, extreme old (Benedict and Root), 389.
ALBERT, A. A. Involutorial Simple Algebras and Real Riemann Matrices, 676.
Algae, blue-green (Sargent), 251.
Algebraic rings (Birkhoff), 571.
Algebra, commutative (Vandiver), 579.
Algebras, Boolean (Stone), 197.
Algebras, involutorial (Albert), 676.
ALLEN, C. E. A Diploid Female Gametophyte of Sphaerocarpos, 147.
ALLEN, E. T. Neglected Factors in the Development of Thermal Springs, 345.
Altmann freezing-drying technique (Goodspeed and Uber), 495.
Ammonium molybdo-tellurates (Donnay and Mélon), 327.
Ammonium nitrate, fused (Audrieth and Schmidt), 221.
Anaerobic decomposition of sulfite waste liquor (Benson and Partansky), 542.
Andromeda nebula (Stebbins and Whitford), 93.
Anisomerism in cranial and dental evolution (Gregory), 1.
Antibodies, blood group, (Keeler and Castle), 465.
Ants, army (Schneirla), 316.
Asymptotic formulae of Riemann and Laplace (Wintner), 57.
Atomic linkages (Rossini), 323.
AUDRIETH, L. F., AND SCHMIDT, M. T. Fused "Onium" Salts as Acids. I. Reactions in Fused Ammonium Nitrate, 221.
Autonomy, absence of (Ephrussi), 420.
Autosomal non-disjunction in Drosophila melanogaster (Stern), 36.
Avena, response to light (Haig), 296, 476.

BAADE, W., AND ZWICKY, F. On Super-Novae, 254.
——. Cosmic Rays from Super-Novae, 259.
BABCOCK, E. B. Genetic Evolutionary Processes, 510.
Bacteria of sea water mud (Benson and Partansky), 542.
BADGER, R. M., AND BARTON, R. C. The Ultra-Violet Absorption Spectrum of Carbon Suboxide Gas, 166.
Balloon, stratosphere (Compton), 79.
BANCROFT, W. D., AND RUTZLER, J. E. Reversible Coagulation in Living Tissue. XII, 501.
Barium II (Saunders, Schneider and Buckingham), 291.
BARTON, R. C. See Badger, R. M., 166.
Basic numbers (Bateman), 63.
BATEMAN, H. Functions Orthogonal in the Hermitian Sense. A New Application of Basic Numbers, 63.
BEANS, H. T. See Dingwall, A., 416.
BENEDICT, F. G., AND ROOT, H. F. The Potentialities of Extreme Old Age, 389.
BENSON, H. K., AND PARTANSKY, A. M. The Rate and Extent of Anaerobic Decomposition of Sulfite Waste Liquor by Bacteria of Sea Water Mud, 542.
BERRY, E. W. Miocene Patagonia, 280.
Beryllium acetate, basic (Pauling and Sherman), 340.
BIRKHOFF, G. Ideals in Algebraic Rings, 571.
BLAKESLEE, A. F. See Cartledge, J. L., 103.
Blood (Fiske), 25.
Blood-groups (Keeler and Castle), 273, 465.
BODINE, J. H. To What Extent Is Developmental Block Dependent upon the Metabolic Activity of the Embryonic Cell?, 640.
Body size of reciprocal hybrids in rabbit crosses (Castle), 621.
BOCHNER, S. Average Distribution of Arbitrary Masses under Group Translations, 206.
BONNER, J. Studies on the Growth Hormone of Plants. V. The Relation of Cell Elongation to Cell Wall Formation, 393.
Boolean algebras (Stone), 197.
Borel's directions (Valiron), 211.
BORSOOK, H., AND KEIGHLEY, G. A Theory of Protein Metabolism in Man, 179.
Brachyury, house mouse (Clark), 276.
Breast, carcinoma of (Dingwall and Beans), 416.
BROCKWAY, L. O. See Pauling, L., 336.
BROWN, C. W., AND HENRY, F. M. The Central Nervous Mechanism for Emotional Responses: II. A Technique for Destroying the Deeper Nuclear Regions within the Cere-

brum with a Minimal Destruction of the Intervening Cortex, 310.
BUCKINGHAM, E. See Saunders, F. A. 291.
Buds, lateral (Skoog and Thimann), 480.
BUERGER, M. J. The Temperature-Structure-Composition Behavior of Certain Crystals, 444.

CAMPBELL, H. L. See Sherman, H. C., 413.
Carbon suboxide gas (Badger and Barton), 166.
Carboxyl group (Pauling and Brockway), 336; (Pauling and Sherman), 340.
Carcinoma of the human breast (Dingwall and Beans), 416.
CARTLEDGE, J. L., AND BLAKESLEE, A. F. Mutation Rate Increased by Aging Seeds as Shown by Pollen Abortion, 103.
CARLEMAN, theorem of (Levinson), 523.
CASTANEDA, M. R. See Zinsser, H., 9.
CASTLE, W. E. Possible Cytoplasmic as Well as Chromosomal Control of Sex in Haploid Males, 101, 219.
——. Body Size of Reciprocal Hybrids in Rabbit Crosses, 621.
——. See Keeler, C. E., 273, 465.
Cell, embryonic (Bodine), 640.
Cell growth (Bonner), 393.
Central nervous mechanism (Brown and Henry), 310.
"Century of Progress" (Compton), 79.
Chained reflexes (Skinner), 234.
Chains, arbitrary (Flexner and Levin), 666.
CHASE, A. M. See Hecht, S., 238.
Chromatophore activator (Kropp and Crozier), 453.
Chromatophores (Parker), 596.
Chromium (Dingwall and Beans), 416.
Chromosomal control of sex in haploid males (Castle), 101, 219.
Chromosome 9 (Creighton), 111.
Chromosome numbers (Siler), 603.
Chromosome sheath (Metz), 159.
Chromosomes in Drosophila (Sturtevant), 515.
Chromosomes in Sciara (Metz and Gay), 617.
Chromosomes, sister sex (Metz), 31.
CLARK, F. H. Linkage Studies of Brachyury (Short Tail) in the House Mouse, 276.
Closed extremals (Morse and Pitcher), 282.
Coagulation, reversible (Bancroft and Rutzler), 501; (Merriam and Rutzler), 608.
COBLENTZ, W. W., AND STAIR, R. Ultra-Violet Transmission Changes in Glass as a Function of the Wave-Length of the Radiation Stimulus, 630.
Coincidence in Drosophila melanogaster (Schweitzer), 154.
Color change in blue-green algae (Sargent), 251.
Coloration, protection (Sumner), 559.
Combinatory logic (Curry), 584.
Commutative algebra (Vandiver), 579.
Commutator machines (Ingram), 23.
Compact continua (Moore), 41.
COMPTON, A. H. Scientific Work in the "Century of Progress" Stratosphere Balloon, 79.
Conformal tensor invariants (Vanderslice), 672.
Conservation of energy (Tolman), 379.
CONSTANCE, L. A Preliminary Revision of the Perennial Species of Eriophyllum, 409.
Continua (Moore), 41.
COOPER, D. C. Development of the Embryo Sac of Lilium Henryi, 163.
Corona, star (Skellett), 461.
Cosmic rays (Epstein), 67; (Baade and Zwicky), 259.
Cosmological models, inhomogeneity on (Tolman), 169.
Cranial evolution among vertebrates (Gregory), 1.
Credonta from Sespe Upper Eocene, California (Stock), 423.
CREIGHTON, H. B. Three Cases of Deficiency in Chromosome of Zea Mays, 111.
Cremona transformations (Huff), 428.
Critical points of harmonic functions (Walsh), 551.
CROZIER, W. J. See Kropp, B., 453.
Crustacean chromatophore activator (Kropp and Crozier), 453.
Crystal structure of basic beryllium acetate (Pauling and Sherman), 340.
Crystals (Buerger), 444.
CURRY, H. B. Functionality in Combinatory Logic, 584.
Cyclic subgroups (Miller), 203, 372.
Cytology, plant (Goodspeed and Uber), 495.
Cytoplasmic control of sex in haploid males (Castle), 101, 219.

DAVENPORT, C. B. Ontogeny and Phylogeny of Man's Appendages, 359.
Decomposition by bacteria (Benson and Partansky), 542.
Deficiency of chromosome 9 in Zea mays (Creighton), 111.
Degenerate quadratic differential forms (Thomas), 215.
Degeneration of nerves (Torrey), 303.
DEMEREC, M. Effect of X-Rays on the Rate of Change in the Unstable Miniature-3 Gene of Drosophila Virilis, 28.
——. Biological Action of Small De-

ficiencies of X-Chromosome of Drosophila melanogaster, 354.
Dental evolution among vertebrates (Gregory), 1.
Depressor substance of blood (Fiske), 25.
Dermatitis, infectious (Wells and ZoBell), 123.
Developing stages of fundulus (Oppenheimer), 536.
Development of rat embryos (Nicholas and Rudnick), 656.
Developmental block (Bodine), 640.
Differential forms (Thomas), 215.
Differentiation of spinors (Veblen and Taub), 85; (Givens), 232.
Diffraction of electrons (Pauling and Brockway), 336.
DINGWALL, A., AND BEANS, H. T. A Spectrographic Study of the Occurrence of Chromium and Molybdenum in Carcinoma of the Human Breast, 416.
Dioecism (Jones), 39.
Diploid female gametophyte of sphaerocarpos (Allen), 147.
Dirac equation (Taub, Veblen and von Neuman), 383.
Discrimination without previous conditioning (Skinner), 532.
Dissection of closed surfaces (Kaufmann and Ursell), 662.
Distribution functions (Haviland), 50.
DONNAY, J. D. H., AND MÉLON, J. Ammonium and Potassium Molybdo-Tellurates, Two Homeomorphous Orthorhombic Substances, 327.
DOOB, J. L. Stochastic Processes and Statistics, 376.
Drosophila melanogaster (Stern), 36; (Schweitzer), 150; (Demerec), 354; (Ephrussi), 420; (Sturtevant), 515.
Drosophila, mutations in (Plough and Ives), 268.
Drosophila virilis (Demerec), 28; (Lebedeff), 613.
DUNN, L. C. A New Gene Affecting Behavior and Skeleton in the House Mouse, 230.

EAST, E. M. Norms of Pollen-Tube Growth in Incompatible Matings of Self-Sterile Plants, 225.
——. The Reaction of the Stigmatic Tissue against Pollen-Tube Growth in Selfed Self-Sterile Plants, 364.
Eggs, development in vitro (Pincus and Enzmann), 121.
Elastic wave analogies (Lindsay), 177.
Electrical commutator machines (Ingram), 23.
Electrons, motion of (Lindsay), 177.
Embryonic cell (Bodine), 640.
Embryo sac of Lilium Henryi (Cooper), 163.
Embryos, rabbit (Waterman), 145; (Keeler and Castle), 273.
Embryos, rat (Nicholas and Rudnick), 656.
Embryos, salamander (Rogers), 247.
EMELEUS, K. G. Notes on Intensities in the Spectrum O II, 115.
Emotional responses (Brown and Henry), 310.
Energies of the atomic linkages (Rossini), 323.
Energy conservation (Tolman), 379.
Energy-momentum principle (Tolman), 437.
Entire functions (Valiron), 211.
ENZMANN, E. V. See Pincus, G., 121.
Eocene (Stock), 150, 349, 423, 518, 625.
EPHRUSSI, B. The Absence of Autonomy in the Development of the Effects of Certain Deficiencies in Drosophila melanogaster, 420.
EPSTEIN, P. S. The Expansion of the Universe and the Intensity of Cosmic Rays, 67.
Equations, integro-differential (Trjitzinsky), 668.
Eriophyllum, perennial species of (Constance), 409.
Evolution, cranial and dental (Gregory), 1.
Evolution of the expanding universe (Lemaitre), 12.
Evolutionary processes (Babcock), 510.
Expanding universe, evolution of (Lemaitre), 12.
Expansion of the universe (Epstein), 67.
Expansion or contraction under gravity (Synge), 635.
Extinction of chained reflexes (Skinner), 234.
Extragalactic nebulae (Hubble and Humason), 264.
Extremals, closed (Morse and Pitcher), 282.
Eye-stock hormone (Kleinholz), 659.

Fishes, dermatitis in (Wells and ZoBell), 123.
FISKE, C. H. The Nature of the Depressor Substance of the Blood, 25.
FLEXNER, W. W., AND LEVIN, M. The Intersection of Arbitrary Chains and Its Boundary, 666.
Force fields (Lindsay), 177.
Force, lines of (Kasner), 130.
Formic acid (Pauling and Brockway), 336.
Fourier series (Hille and Tamarkin), 369.
FRED, E. B., AND WILSON, P. W. On Photosynthesis and Free Nitrogen Assimilation by Leguminous Plants, 403.
Freezing-drying technique, Altmann (Goodspeed and Uber), 495.
Functionality in logic (Curry), 584.
Functions, distribution (Haviland), 50.

Functions, entire (Valiron), 211.
Functions, harmonic (Walsh), 551.
Functions, orthogonal in the Hermitian sense (Bateman), 63.
Fundulus, early developing stages (Oppenheimer), 536.
Fused "onium" salts as acids (Audrieth and Schmidt), 221.

Gametophyte, female (Allen), 147.
Gas, carbon suboxide (Badger and Barton), 166.
GAY, E. H. See Metz, C. H., 617.
Generals, resolution into (Wilson), 193.
Genetic evolutionary processes (Babcock), 510.
Genes of mouse (Dunn), 230.
Geocentric distance (Sciobereti) ,539.
Geodesics (Hedlund), 136.
GIVENS, J. W., JR. Projective Differentiation of Spinors, 232.
Glass, changes in transmission of (Coblentz and Stair), 630.
GOLDSMITH, E. D. Correlation in Planarian Regeneration, 555.
GOODSPEED, T. H., AND UBER, F. M. Application of the Altmann Freezing-Drying Technique to Plant Cytology, 495.
Grafts of spinal cord (Rogers), 247.
Gravity (Synge), 635.
GREGORY, W. K. Polyisomerism and Anisomerism in Cranial and Dental Evolution among Vertebrates, 1.
Groups (Miller), 127, 203, 243, 288, 372, 430.
Groups (Smith), 577.
Groups, Abelian (van Kampen), 434.
Group translations (Bochner), 206.
Growth (Sherman and Campbell), 413.
Growth hormone (Skoog and Thimann), 480.
Growth hormone of plants (Bonner), 393.
Growth, motion of (Wetzel), 183.

HAIG, C. The Effect of Intensity and Wave-Length on the Response of Avena to Light, 296.
——. The Spectral Sensibility of Avena, 476.
Hair color of mammals (Sumner), 397.
Haploid males (Castle), 101, 219.
Haploids in hordeum vulgare (Johansen), 98.
Harmonic functions (Walsh), 551.
HAVILAND, E. K. On Distribution Functions and Their Laplace-Fourier Transforms, 50.
Heat induced mutations in Drosophila (Plough and Ives), 268.
HECHT, S. A Theoretical Basis for Intensity Discrimination in Vision, 644.
HECHT, S., AND CHASE, A. M. Anomalies in the Absorption Spectrum of Visual Purple, 238.
HEDLUND, G. A. On the Metrical Transitivity of the Geodesics on a Surface of Constant Negative Curvature, 136.
HENDERSON, M. C. See Livingston, M. S., 470.
HENRY, F. W. See Brown, C. W., 310.
Hermaphroditism in Drosophila virilis (Lebedeff), 613.
Hermitian orthogonal functions (Bateman), 63.
Heterotopic spinal cord grafts (Rogers), 247.
HILLE, E., AND TAMARKIN, J. D. On the Theory of Laplace Integrals. II, 140.
——. On the Summability of Fourier Series. Fifth Note, 369.
Hordeum vulgare, haploids in (Johansen), 98.
Hormone, eye-stock (Kleinholz), 659.
Hormone, growth (Bonner), 393; (Skoog and Thimann), 480.
HOVGAARD, W. An Investigation of the Stresses in Longitudinal Welds, 17.
HUBBLE, E., AND HUMASON, M. L. The Velocity-Distance Relation for Isolated Extragalactic Nebulae, 264.
HUFF, G. B. A Note on Cremona Transformations, 428.
HUMASON, M. L. See Hubble, E., 264.
Hybrids, reciprocal (Castle), 621.
Hydrocarbons, paraffin (Rossini), 323.
Hyopsodontidæ (Stock), 349.
Hypertragulid (Stock), 625.

Ideals in algebraic rings (Birkhoff), 571.
Immunization in typhus fever (Zinsser and Castaneda), 9.
Indeterminacy principle (White), 525.
INGRAM, W. H. On the Dynamical Theory of Electrical Commutator Machines, 23.
Inhibition of lateral buds (Skoog and Thimann), 480.
Inhomogeneity of cosmological models (Tolman), 169.
Instability (Morse), 46.
Integro-differential equations (Trjitzinsky), 668.
Intensity discrimination in vision (Hecht), 644.
Intersection of arbitrary chains (Flexner and Levin), 666.
Invariants (Morse and Pitcher), 282.
Invariants, conformal tensor (Vanderslice), 672.
Involutorial algebras (Albert), 676.
Integrals, Laplace (Hille and Tamarkin), 140.
Intensities in the spectrum O II (Emeleus), 115.

Interference in Drosophila melanogaster (Schweitzer), 154.
Isomorphic subgroups (Miller), 430.
IVES, P. T. See Plough, H. H., 268.

JOHNSON, M. H., JR. On the Vector Model for Almost Closed Shells, 117.
JOHANSEN, D. A. Haploids in Hordeum Vulgare, 98.
JONES, D. F. Unisexual Maize Plants and Their Relation to Dioecism in Other Organisms, 39.

KAPLAN, N. V_3 in R_6, 489.
——. V_3 in R_6 of $[1,2]_a$, 574.
KASNER, E. General Theorems on Trajectories and Lines of Force, 130.
KAUFMANN, B., AND URSELL, H. D. The Dissection of Closed Surfaces and the Phragmen-Brouwer-Alexandroff Theorem, 662.
KEELER, C. E., AND CASTLE, W. E. Blood-Group Incompatibility in Rabbit Embryos and in Man, 273.
——. The Influence of Pregnancy upon the Titre of Immune (Blood Group) Antibodies in the Rabbit, 465.
KEIGHLEY, G. See Borsook, H., 179.
KIMBALL, G. E. See Shortley, G. H., 82.
KLEINHOLZ, L. H. Eye-Stalk Hormone and the Movement of Distal Retinal Pigment in Palaemonotes, 659.
KROPP, B., AND CROZIER, W. J. The Production of the Crustacean Chromatophore Activator, 453.

Laplace, asymptotic formulae of (Wintner), 57.
Laplace-Fourier transforms (Haviland), 50.
Laplace integrals (Hille and Tamarkin), 140.
Laplace-Leuschner method for parabolic orbits (Sciobereti), 539.
LAWRENCE, E. O. See Livingston, M. S., 470.
Least squares (Sterne), 565, 601.
LEBEDEFF, G. A. Genetics of Hermaphroditism in Drosophila virilis, 613.
Leguminous plants (Fred and Wilson), 403.
LEMAITRE, G. Evolution of the Expanding Universe, 12.
LEVIN, M. See Flexner, W. W., 666.
LEVINSON, N. On a Theorem of Carleman, 523
Light-response of Avena (Haig), 296, 476.
Lilium Henryi (Cooper), 163.
LINDSAY, R. B. Elastic Wave Analogies to the Motion of Electrons in Force Fields, 177.
Lines of force (Kasner), 130.
Linkage studies in house mouse (Clark), 276.
LIVINGSTON, M. S., HENDERSON, M. C., AND LAWRENCE, E. O. Radioactivity Artificially Induced by Neutron Bombardment, 470.
Localization of pure tones (Stevens and Newman), 593.
Logic (Curry), 584.
Longitudinal welds, stresses in (Hovgaard), 17.

Machines, commutator (Ingram), 23.
Maize plants (Jones), 39.
Mammalian eggs (Pincus and Enzmann), 121.
Mammals, hair color of (Sumner), 397.
Man, blood-group incompatibility in (Keeler and Castle), 273.
Man, theory of protein metabolism in (Borsook and Keighley), 179.
Man's appendages (Davenport), 359.
Manifolds (Smith), 577.
MARIA, A. J. The Potential of a Positive Mass and the Weight Function of Wiener, 485.
Mass, positive (Maria), 485.
Masses, distribution of (Bochner), 206.
Matings of self-sterile plants (East), 225.
Matrices, Riemann (Albert), 676.
Mechanics, non-conservative (Tolman), 437, 439.
Mechanics, quantum (Shortley and Kimball), 82.
MÉLON, J. See Donnay, J. D. H., 327.
MERRIAM, H. E., AND RUTZLER, J. E., JR. Reversible Coagulation in Living Tissue. XIII, 608.
Metabolic activity of embryonic cell (Bodine), 640.
Metabolism, protein (Borsook and Keighley), 179.
Metric in non-conservative mechanics (Tolman), 437, 439.
Metrical transitivity of the geodesics (Hedlund), 136.
METZ, C. W. Evidence Indicating That in Sciara the Sperm Regularly Transmits Two Sister Sex Chromosomes, 31.
——. The Rôle of the "Chromosome Sheath" in Mitosis, and Its Possible Relation to Phenomena of Mutation, 159.
METZ, C. W., AND GAY, E. H. Internal Organization of Salivary Gland Chromosomes of Sciara in Relation to Genes, 617.
Microsyopsinæ (Stock), 349.
MILLER, G. A. Minimum Number of Squares in a Group When Not All of Them Are Relatively Commutative, 127.
——. Groups Whose Squares Constitute Cyclic Subgroups, 203.

——. Groups Involving Three and Only Three Squares, 243.
——. Confusions in the Use of the Mathematical Term Group, 288.
——. Groups of Order 2^n Whose Squared Elements Constitute a Cyclic Subgroup, 372.
——. Distinct Groups Whose Subgroups Are Simply Isomorphic, 430.
Miniature-3 gene in Drosophila virilis (Demerec), 28.
Miocene Patagonia (Berry), 280.
Mitosis (Metz), 159.
Molybdenum (Dingwall and Beans), 416.
Molybdo-tellurates (Donnay and Mélon), 327.
MOORE, R. L. Concerning Compact Continua Which Contain No Continuum That Separates the Plane, 41.
MORSE, M. Does Instability Imply Transitivity?, 46.
MORSE, M., AND PITCHER, E. On Certain Invariants of Closed Extremals, 282.
Motion of growth (Wetzel), 183.
Mouse, house (Dunn), 230; (Clark), 276.
Multiplet strengths (Shortley), 591.
Mutation (Metz), 159.
Mutation rate increased by aging seeds (Cartledge and Blakeslee), 103.
Mutations in Drosophila (Plough and Ives), 268.

Nebula, Andromeda (Stebbins and Whitford), 93.
Nebulae, extragalactic (Hubble and Humason), 264.
Nerve degeneration, temperature coefficient of (Torrey), 303.
Nerves, momentarily stimulated (Parker), 306.
NEWMAN, E. B. See Stevens, S. S., 593.
Neutron bombardment (Livingston, Henderson and Lawrence), 470.
NICHOLAS, J. S., AND RUDNICK, D. The Development of Rat Embryos in Tissue Culture, 656.
Nitrate, fused ammonium (Audrieth and Schmidt), 221.
Nitrogen assimilation by legumes (Fred and Wilson), 403.
Non-conservative mechanics (Tolman), 437, 439.
Novae (Baade and Zwicky), 254, 259.
Nuclear regions, destruction of (Brown and Henry), 310.
Numbers, basic (Bateman), 63.

Old age (Benedict and Root), 389.
"Onium" salts (Audrieth and Schmidt), 221.
Ontogeny of man's appendages (Davenport), 359.
OPPENHEIMER, J. M. Experiments on Early Developing Stages of Fundulus, 536.
Orbits, parabolic (Sciobereti), 539.
Oredont skeleton (Stock), 518.
Orthogonal functions (Bateman), 63.
Oxygen, O II, spectrum (Emeleus), 115.

Palaemonetes (Kleinholz), 659.
Parabolic orbits (Sciobereti), 539.
Paraffin hydrocarbons (Rossini), 323.
PARKER, G. H. The Prolonged Activity of Momentarily Stimulated Nerves, 306.
——. Acetyl Choline and Chromatophores, 596.
PARTANSKY, A. M. See Benson, H. K., 542.
Patagonia, miocene (Berry), 280.
PAULING, L., AND BROCKWAY, L. O. The Structure of the Carboxyl Group. I. The Investigation of Formic Acid by the Diffraction of Electrons, 336.
PAULING, L., AND SHERMAN, J. The Structure of the Carboxyl Group. II. The Crystal Structure of Basic Beryllium Acetate, 340.
Perennial species of eriophyllum (Constance), 409.
Photosynthesis (Fred and Wilson), 403.
Phragmen-Brouwer-Alexandroff theorem (Kaufmann and Ursell), 662.
Phylogeny of man's appendages (Davenport), 359.
PINCUS, G., and ENZEMANN, E. V. Can Mammalian Eggs Undergo Normal Development in Vitro?, 121.
PITCHER, E. See Morse, M., 282.
Planarian regeneration (Goldsmith), 555.
Plants, self-sterile (East), 225, 364.
PLOUGH, H. H., AND IVES, P. T. Heat Induced Mutations in Drosophila, 268.
Pollen abortion (Cartledge and Blakeslee), 103.
Pollen-tube growth (East), 225, 364.
Polyisomerism in cranial and dental evolution (Gregory), 1.
Potential of a positive mass (Maria), 485.
Potassium molybdo-tellurates (Donnay and Mélon), 327.
Potentialities of extreme old age (Benedict and Root), 389.
Pregnancy and immunity (Keeler and Castle), 465.
Primate, eocene (Stock), 150.
Projective differentiation of spinors (Veblen and Taub), 85; (Givens), 232.
Projective relativity (Taub, Veblen and von Neuman), 383.
Prolonged activity of momentarily stimulated nerves (Parker), 306.
Protective coloration (Sumner), 559.
Protein metabolism in man (Borsook and Keighley), 179.
Purple, visual (Hecht and Chase), 238.

Quadratic differential forms (Thomas), 215.
Quantum mechanics (Shortley and Kimball), 82.

R_6 (Kaplan), 489, 574.
Rabbits (Keeler and Castle), 273, 465; (Castle), 621.
Rabbit embryos (Waterman), 145.
Radioactivity (Livingston, Henderson and Lawrence), 470.
Raiding in army ants (Schneirla), 316.
Rat embryos (Nicholas and Rudnick), 656.
Rays, cosmic (Epstein), 67; (Baade and Zwicky), 259.
Reflexes, chained (Skinner), 234.
Regeneration, planarian (Goldsmith), 555.
Relativity, projective (Taub, Veblen and von Neuman), 383.
Resolution of tests (Wilson and Worcester), 189; (Wilson), 193.
Responses, emotional (Brown and Henry), 310.
Reversible coagulation (Bancroft and Rutzler), 501; (Merriam and Rutzler), 608.
Ricciaceae, chromosome numbers in (Siler), 603.
Riemann, asymptotic formulae of (Wintner), 57.
Riemann matrices (Albert), 676.
Rings, algebraic (Birkhoff), 571.
Ritz method, modified (Weinstein), 529.
ROGERS, W. M. Heterotopic Spinal Cord Grafts in Salamander Embryos, 247.
ROOT, H. F. See Benedict, F. G., 389.
ROSSINI, F. D. The Energies of the Atomic Linkages in the Normal Paraffin Hydrocarbons, 323.
RUDNICK, D. See Nicholas, J. S., 656.
RUTZLER, JR., J. E. See Bancroft, W. D., 501.
——. See Merriam, H. E., 608.

Salamander embryos (Rogers), 247.
Salivary glands (Metz and Gay), 616.
Salts, fused "onium" (Audrieth and Schmidt), 221.
SARGENT, M. C. Causes of Color Changes in Blue-Green Algae, 251.
SAUNDERS, F. A., SCHNEIDER, E. G., AND BUCKINGHAM, E. The Strontium II and Barium II Spectra, 291.
SCHMIDT, M. T. See Audrieth, L. F., 221.
SCHNEIDER, E. G. See Saunders, F. A., 291.
SCHNEIRLA, T. C. Raiding and Other Outstanding Phenomena in the Behavior of Army Ants, 316.
SCHWEITZER, M. D. Coincidence and Interference in Drosophila melanogaster, 154.
Sciara (Metz), 31; (Metz and Gay), 617.
SCIOBERETI, R. H. On the Determination of the Geocentric Distance in the Laplace-Leuschner Direct Method for Parabolic Orbits, 539.
Seeds, aging and mutation (Cartledge and Blakeslee), 103.
Segregation of chromosomes (Sturtevant), 515.
Self-sterile plants (East), 225, 364.
Sespe of California (Stock), 349, 423, 518, 625.
Sex, control of (Castle), 101, 219.
Shells, almost closed (Johnson), 117.
SHERMAN, H. C., AND CAMPBELL, H. L. Observations upon Growth from the Viewpoint of Statistical Interpretation, 413.
SHERMAN, J. See Pauling, L., 340.
SHORTLEY, G. H. The Calculation of Relative Multiplet Strengths in a Transition Array, 591.
SHORTLEY, G. H., AND KIMBALL, G. E. Analysis of Non-Commuting Vectors with Application to Quantum Mechanics and Vector Calculus, 82.
SILER, M. B. Chromosome Numbers in Certain Ricciaceae, 603.
Sister sex chromosomes (Metz), 31.
Skeleton in house mouse (Dunn), 230.
SKELLETT, A. M. Proposal of a Method of Observing the Solar Corona without an Eclipse, 461.
SKINNER, B. F. The Extinction of Chained Reflexes, 234.
——. A Discrimination without Previous Conditioning, 532.
SKOOG, F., AND THIMANN, K. V. Further Experiments on the Inhibition of the Development of Lateral Buds by Growth Hormone, 480.
SMITH, P. A. The Fundamental Group of a Group Manifold, 577.
Solar corona (Skellett), 461.
Specifics, resolution into (Wilson), 193.
Spectra, Barium II and Strontium II (Saunders, Schneider and Buckingham), 291.
Spectral sensibility of avena (Haig), 476.
Spectrum absorption (Badger and Barton), 166.
Spectrum O II, intensities of (Emeleus), 115.
Spectrum, visual purple (Hecht and Chase), 238.
Sphaerocarpos (Allen), 147.
Spinal cord grafts (Rogers), 247.
Spinors (Veblen and Taub), 85; (Givens), 232.
Springs, thermal (Allen), 345.
Squares (Miller), 127, 203, 243.
STAIR, R. See Coblentz, W. W., 630.
Standard deviation (Sterne), 601.
Statistics (Doob), 376.
STEBBINS, J., AND WHITFORD, A. E. The

Diameter of the Andromeda Nebula, 93.
STERN, C. On the Occurrence of Translocations and Autosomal Non-Disjunction in Drosophila melanogaster, 36.
STERNE, T. E. The Accuracy of Least Squares, 565.
——. The Accuracy of Least Squares Solutions. Part II: The Standard Deviation of the Errors of Linear Equations of Condition, 601.
STEVENS, S. S. The Attributes of Tones, 457.
STEVENS, S. S., AND NEWMAN, E. B. The Localization of Pure Tones, 593.
Stigmatic tissue (East), 364.
Stochastic processes and statistics (Doob), 376.
STOCK, C. A Second Eocene Primate from California, 150.
——. Microsyopsinæ and Hyopsodontidæ in the Sespe Upper Eocene, California, 349.
——. New Credonta from the Sespe Upper Eocene, California, 423.
——. On the Occurrence of an Oreodont Skeleton in the Sespe of South Mountain, California, 518.
——. A Hypertragulid from the Sespe Uppermost Eocene, California, 625.
STONE, M. H. Boolean Algebras and Their Application to Topology, 197.
Stratosphere balloon (Compton), 79.
Stresses in longitudinal welds (Hovgaard), 17.
Strontium II (Saunders, Schneider and Buckingham), 291.
Structure of the carboxyl group (Pauling and Brockway), 336; (Pauling and Sherman), 340.
STURTEVANT, A. H. Preferential Segregation of the Fourth Chromosomes in Drosophila melanogaster, 515.
Summability of Fourier series. Fifth Note (Hille and Tamarkin), 369.
SUMNER, F. B. A Test of the Possible Effects of Visual Stimuli upon the Hair Color of Mammals, 397.
——. Does "Protective Coloration" Protect? Results of Some Experiments with Fishes and Birds, 559.
Super-novae (Baade and Zwicky), 254, 259.
Surface of constant negative curvature (Hedlund), 136.
Surfaces, closed (Kaufmann and Ursell), 662.
Survival of young rabbit embryos on artificial media (Waterman), 145.
SYNGE, J. L. On the Expansion or Contraction of a Symmetrical Cloud under the Influence of Gravity, 635.
TAMARKIN, J. D. See Hille, E., 140, 369.
TAUB, A. H. See Veblen, O., 85.
TAUB, A. H., VEBLEN, O., AND VON NEUMAN, J. The Dirac Equation in Projective Relativity, 383.
Temperature-structure-composition behavior of certain crystals (Buerger), 444.
Tensor invariants (Flexner and Levin), 666.
Tests, resolution of four (Wilson and Worcester), 189.
Theoretical basis for intensity discrimination in vision (Hecht), 644.
Thermal springs (Allen), 345.
THIMANN, K. V. See Skoog, F., 480.
THOMAS, T. Y. The Reduction of Degenerate Quadratic Differential Forms, 215.
TOLMAN, R. C. Effect of Inhomogeneity on Cosmological Models, 169.
——. Remarks on the Possible Failure of Energy Conservation, 379.
——. Suggestions as to the Energy-Momentum Principle in a Non-Conservative Mechanics, 437.
——. Suggestions as to Metric in a Non-Conservative Mechanics, 439.
Tones (Stevens), 457; (Stevens and Newman), 593.
Topology and Boolean algebras (Stone), 197.
TORREY, T. W. Temperature Coefficient of Nerve Degeneration, 303.
Trajectories, general theorems on (Kasner), 130.
Transformations, cremona (Huff), 428.
Transforms, Laplace-Fourier (Haviland), 50.
Transition array (Shortley), 591.
Transitivity (Morse), 46; (Hedlund), 136.
Translations, group (Bochner), 206.
Translocations in Drosophila melanogaster (Stern), 36.
Transmission, ultra-violet (Coblentz and Stair), 630.
TRJITZINSKY, W. J. The General Case of Linear Integro-Differential Equations, 668.
Typhus fever (Zinsser and Castaneda), 9.

UBER, F. M. See Goodspeed, T. H., 495.
Ultra-violet absorption spectrum (Badger and Barton), 166.
Ultra-violet transmission (Coblentz and Stair), 630.
Unisexual maize plants (Jones), 39.
Universe, expanding (Lemaitre), 12.
Universe, expansion of (Epstein), 67.
URSELL, H. D. See Kaufmann, B., 662.

V_3 (Kaplan), 485, 574.
V_3 in R_4 of $[1,2]_a$ (Kaplan), 574.

VALIRON, G. Entire Functions and Borel's Directions, 211.
VANDERSLICE, J. L. Conformal Tensor Invariants, 672.
VANDIVER, H. S. On the Foundations of a Constructive Theory of Discrete Commutative Algebra, 579.
VAN KAMPEN, E. R. Locally Compact Abelian Groups, 434.
VEBLEN, O., AND TAUB, A. H. Projective Differentiation of Spinors, 85.
VEBLEN, O. See Taub, A. H., 383.
Vector calculus (Shortley and Kimball), 82.
Vector model for almost closed shells (Johnson), 117.
Velocity-distance relation (Hubble and Humason), 264.
Vertebrates, cranial and dental evolution (Gregory), 1.
Vision (Hecht), 644.
Visual purple (Hecht and Chase), 238.
Visual stimuli (Sumner), 397.
VON NEUMAN, J. See Taub, A. H., 283.

WALSH, J. L. Note on the Location of the Critical Points of Harmonic Functions, 551.
WATERMAN, A. J. Survival of Young Rabbit Embryos on Artificial Media, 145.
WEINSTEIN, D. H. Modified Ritz Method, 529.
Welds, longitudinal (Hovgaard), 17.
WELLS, N. A., AND ZOBELL, C. E. Achromobacter Ichthyodermis, N. Sp., the Etiological Agent of an Infectious Dermatitis of Certain Marine Fishes, 123.
WETZEL, N. C. On the Motion of Growth, 183.
WHITE, F. E. Some Special Cases of the Indeterminacy Principle, 525.
WHITFORD, A. E. See Stebbins, J., 93.
Wiener, weight function of (Maria), 485.
WILSON, E. B. On Resolution into Generals and Specifics, 193.
WILSON, E. B., AND WORCESTER, J. The Resolution of Four Tests, 189.
WILSON, P. W. See Fred, E. B., 403.
WINTNER, A. On the Asymptotic Formulae of Riemann and of Laplace, 57.
WORCESTER, J. See Wilson, E. B., 189.

X-chromosomes (Demerec), 354.
X-rays, effect on Drosophila virilis (Demerec), 28.

Zea mays (Creighton), 111.
ZINSSER, H., AND CASTANEDA, M. R. Active and Passive Immunization in Typhus Fever, 9.
ZOBELL, C. E. See Wells, N. A., 123.
ZWICKY, F. See Baade, W., 254, 259.

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